Copulas Related to Piecewise Monotone Functions of the Interval and Associated Processes

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Abstract

In this work we derive the copulas related to vectors coming from the so-called chaotic stochastic processes. These are defined by the iteration of certain piecewise monotone functions of the interval $[0,1]$ to some initial random variable. We study some of its properties and present some examples. Since often these type of copulas do not have closed formulas, we provide a general method of approximation which converges uniformly to the true copula. Our results cover a wide class of processes, including the so-called Manneville-Pomeau processes. The general theory is applied to the parametric estimation in certain chaotic processes and we also present a Monte Carlo study.

Keywords. Chaotic Processes, Copulas, Interval Maps, Invariant Measures, Parametric Estimation.

1 Introduction

Let $T:[0,1]\to[0,1]$ be a smooth function of the interval $[0,1]$ and suppose there exists an absolutely continuous T-invariant probability measure, say, μ_T. Let U_0 be distributed as μ_T. We can define a stochastic process by setting $X_t := \varphi\big(T^t(U_0)\big)$, $t\in\mathbb{N}$, for a given μ_T-integrable function $\varphi:[0,1]\to\mathbb{R}$. This type of stochastic process has been applied in a variety of problems from rock drilling (see Lasota and Mackey, 1994 and references therein) to intermittency in human cardiac rate (see Zebrowsky, 2001). Realizations of this type of process usually present complex dynamics, chaotic behavior and instability with respect to the initial point X_0.

In this work we study the copulas related to random vectors obtained from these type of stochastic process. More specifically, we are interested in the case where the transformation T is a piecewise monotone function of the interval and φ is a strictly monotone function. Our results cover a wide class of stochastic process, such as the so-called Manneville-Pomeau process (studied in Lopes and Pumi, 2013) and the process related to the tent transformation (see Example 6.2) among many others (see, for instance, Lasota and Mackey, 1994).

The present work greatly generalizes the results in Lopes and Pumi (2013), where the authors derive and study the copulas related to Manneville-Pomeau processes, by considering T belonging to a certain general class of piecewise smooth transformations which can have either increasing, decreasing, or monotone full branches. We consider the important bidimensional case as well as the multidimensional case. As the problem of existence of an invariant measure for a given transformation is usually a difficult one, often the copulas derived do not have closed formulas. In that case, we have to rely on approximations to study these copulas. In this direction, we develop a somewhat general approximation to the copulas which is shown to converge uniformly to the theoretical one. The problem of random variate generation of the copulas presented here is also addressed. As an application of the general theory, the problem of parametric estimation in certain chaotic process is discussed and an estimation procedure moti-

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vated by the results obtained in this paper is proposed. To assess the finite sample performance of the proposed estimator, a Monte Carlo study is performed. Examples are also provided.

The paper is organized as follows: in the next section, we briefly review some concepts and results on copulas and introduce the class of stochastic processes we will be interested in this work. Section 3 is devoted to determine the copulas related to any pair (X_t, X_{t+h}) coming from the aforementioned processes and to explore some consequences. In section 4, some multidimensional extensions are shown. In Section 5 we develop a general approximation to the copulas derived in Section 3 and prove its uniform convergence to the true copula. Random variate generation is also addressed. Section 6 brings two examples. In Section 7 we apply the general theory of Section 3 to the problem of parametric estimation in certain chaotic processes. A Monte Carlo study is also performed. The conclusions are reserved to Section 8.

2 Preliminaries

We begin by establishing the necessary mathematical framework. We say that a map $T : I \to I$, where $I := [0, 1]$, is of class $C^{1+\alpha}_{1\text{-}1}$ if it is a one-to-one function of class $C^{1+\alpha}$ (the class of C^1 functions whose derivative is α-Hölder continuous), for $\alpha \in (0, 1)$. A map $T : I \to I$, is said to be *finitely piecewise* $C^{1+\alpha}_{1\text{-}1}$ if there exists a partition $\{I_k\}_{k=1}^{s}$ of I such that the restriction of T to the interior of each I_k is a $C^{1+\alpha}_{1\text{-}1}$ function. Each I_k will be called a *node* of T. If, in addition, the restriction of T to the interior of each I_k is monotone (increasing or decreasing), then we call T a *finitely piecewise monotone* (*increasing* or *decreasing*) $C^{1+\alpha}_{1\text{-}1}$ function.

For consistency, when we call for a partition relative to a map T, we shall always mean the *maximal partition* $\{I_k\}$, in the sense that if $\{\tilde{I}_k\}$ is another partition such that T restricted to the interior of each $\tilde{I}_k$ is a $C^{1+\alpha}_{1\text{-}1}$ function, then, for each k, there exists j such that $\tilde{I}_k \subseteq I_j$. We shall refer as the *net* relative to the nodes of a (finitely piecewise $C^{1+\alpha}_{1\text{-}1}$) map T, the collection of all endpoints of each I_k in the maximal partition $\{I_k\}$. Notice that the net of the nodes of T are discontinuity points of T, except, perhaps, the points 0 and 1. For the sake of simplicity, but without loss of generality, we shall assume that $I_k = [a_{k-1}, a_k)$ and that T is right continuous, except at 1 where we assume that T is left continuous.

Given an arbitrary function $T : I \to I$ one can ask whether there exists a smooth T-invariant probability measure. The problem of establishing the existence of such measure is often a hard one. The literature on the subject is relatively extense and has a long history. Rényi (1957) shows that, for transformation of the form $T_a(x) = ax \pmod 1$, a unique absolutely continuous invariant probability measure always exists. Remember that a map $T : I \to I$ is said to be *uniformly expanding* in I if $|T'(x)| \geq \kappa > 1$, for all $x \in I$, whenever T' is defined. Lasota and Yorke (1973) show that for piecewise smooth uniformly expanding function there always exists an absolutely continuous invariant measure. Under more stringent, but easily verifiable conditions, Pianigiani (1980) show that this measure is unique and it is a probability measure. However, many non-expanding functions are known to possess such a measure. Sufficient conditions for this to happen for non-expanding functions can be found, for instance, in Bowen (1979), Pianigiani (1980) and Pianigiani (1981). We shall denote the space of all transformations $T : I \to I$ for which an absolutely continuous invariant probability measure exists by $\mathscr{S}$. More specifically, let us define the following spaces:

$$\mathcal{T}^{\updownarrow} := \big\{T \in \mathscr{S} \text{ and } T \text{ is finitely piecewise monotone } C^{1+\alpha}_{1\text{-}1} \text{ function}\big\};$$
$$\mathcal{T}^{\downarrow} := \big\{T \in \mathscr{S} \text{ and } T \text{ is finitely piecewise decreasing } C^{1+\alpha}_{1\text{-}1} \text{ function}\big\};$$
$$\mathcal{T}^{\uparrow} := \big\{T \in \mathscr{S} \text{ and } T \text{ is finitely piecewise increasing } C^{1+\alpha}_{1\text{-}1} \text{ function}\big\}.$$

Certainly $\mathcal{T}^{\uparrow} \bigcap \mathcal{T}^{\downarrow} = \emptyset$ and $\mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow} \subsetneq \mathcal{T}^{\updownarrow}$. Also notice that if $T \in \mathcal{T}^{\updownarrow}$, T has only full branches in the sense that $T : I_k \to [0, 1)$ is onto. When $T \in \mathcal{T}^{\updownarrow}$ and has $s > 1$ nodes, for $t = 1, \cdots, s$, we shall define

$$K_t^{\uparrow} := \{k : \ T^t|_{I_k} \text{ is increasing}\} \qquad \text{and} \qquad K_t^{\downarrow} := \{k : \ T^t|_{I_k} \text{ is decreasing}\}, \tag{2.1}$$

where, as usual, $T^k = T \circ T^{k-1}$ denotes the k-fold composition. Clearly, $K_t^{\uparrow} \bigcap K_t^{\downarrow} = \emptyset$, $K_t^{\uparrow} = \emptyset$, if $T \in \mathcal{T}^{\downarrow}$ and $K_t^{\downarrow} = \emptyset$, if $T \in \mathcal{T}^{\uparrow}$.

In this paper we are interested in studying the copulas related to the following class of stochastic process.

Definition 2.1. Let $T \in \mathcal{T}^{\updownarrow}$ and let μ_T be a T-invariant probability measure. Let U_0 be a random variable distributed according to μ_T and $\varphi : I \to \mathbb{R}$ be a function in $\mathcal{L}^1(\mu_T)$. The stochastic process given by

$$X_t := (\varphi \circ T^t)(U_0), \quad \text{for all } t \in \mathbb{N}, \tag{2.2}$$

is called a $\mathcal{T}_\varphi^{\updownarrow}$-*induced process* (or $\mathcal{T}_\varphi^{\updownarrow}$ process, for short).

By taking, in Definition 2.1, $T \in \mathcal{T}^{\uparrow}$ or $T \in \mathcal{T}^{\downarrow}$ instead, we define a $\mathcal{T}_\varphi^{\uparrow}$-*induced process* and a $\mathcal{T}_\varphi^{\downarrow}$-*induced process*, respectively, or, for short, $\mathcal{T}_\varphi^{\uparrow}$ and $\mathcal{T}_\varphi^{\downarrow}$ process. The processes just defined are stationary since μ_T is T-invariant and absolutely continuous. the relationshipe betweem=n X_t and U_0 is sometimes called mutual complete dependence as, for instance, in Siburg and Stoimev (2010).

Regarding copulas, a n-dimensional copula is a distribution function whose marginals are uniformly distributed on I and whose support is the n-dimensional cube I^n. The usefulness of copulas lies on its ability to model dependence independently of the marginals and vice-versa. The literature on the subject has grown enormously, especially in the last decade, due to the discover of several applications of copulas on many areas such as finance, actuarial science, time series, hydrology, among others. For instance, an interesting application of copulas in finances can be found in Wang et al. (2009), where the authors present one-factor models to pricing credit default index swap tranches and collateralized debt obligations, based on heavy tailed copulas especially designed to allow a continuous time tail-fatness control. We refer the reader to Cherubini et al. (2004), McNeil et al. (2005) and references therein for applications in finances, Frees and Valdez (1998) for a review on the use of copula on actuarial sciences, Chen and Fan (2006) for an interesting copula based on a unidimensional time series model and Salvadori et al, (2007) for applications of copulas in hydrology.

Among others, the invariance by almost everywhere increasing function and the simple functional form the copula takes when the transformation is decreasing almost everywhere are some of the properties we shall use very often in what follows. In the next theorem, we summarize these properties. The proof can be found in Nelsen (2006). In this work the measure implicit to phrases like “almost everywhere” and “almost sure” will be the appropriate Lebesgue measure.

Theorem 2.1. *Let C be any copula and let $f_1, \cdots, f_n$ be almost everywhere increasing functions. Then $C_{f_1(X_1),\cdots,f_n(X_n)}(u_1, \cdots, u_n) = C_{X_1,\cdots,X_n}(u_1, \cdots, u_n)$. Moreover, if f and g are two almost everywhere decreasing functions instead, then $C_{f(X),g(Y)}(u, v) = u + v - 1 + C_{X,Y}(1 - u, 1 - v)$.*

The next theorem, the so-called Sklar's theorem, is the key result for copulas. See Schweizer and Sklar (2005) for a sketch of the proof in the n-dimensional case, Durante et al. (2012, 2013) for alternative proofs and Nelsen (2006) for a more detailed proof in the bidimensional case.

Theorem 2.2 (Sklar's Theorem)**.** *Let $X_1, \cdots, X_n$ be random variables with joint distribution function H and marginals $F_1, \cdots, F_n$, respectively. Then, there exists a copula C such that,*

$$H(x_1, \cdots, x_n) = C\big(F_1(x_1), \cdots, F_n(x_n)\big), \quad \textit{for all } \ (x_1, \cdots, x_n) \in \mathbb{R}^n.$$

If the F_i's are continuous, then C is unique. Otherwise, C is uniquely determined on $\mathrm{Ran}(F_1) \times \cdots \times \mathrm{Ran}(F_n)$, *where* $\mathrm{Ran}(F)$ *denotes the range of the function F. The converse also holds. Furthermore,*

$$C(u_1, \cdots, u_n) = H\big(F_1^{(-1)}(u_1), \cdots, F_n^{(-1)}(u_n)\big), \quad \text{for all } (u_1, \cdots, u_n) \in I^n,$$

where for a function F, $F^{(-1)}$ denotes its pseudo-inverse given by $F^{(-1)}(x) := \inf\big\{u \in \mathrm{Ran}(F) : F(u) \geq x\big\}$.

For more details on the theory of copulas we refer the reader to Joe (1997) and Nelsen (2006). Copulas are also in close connection to probabilistic metric spaces. See Schweizer and Sklar (2005) for details on this matter.

3 Bidimensional case

In this section we shall investigate the bidimensional copulas associated to $\mathcal{T}_\varphi^\uparrow$, $\mathcal{T}_\varphi^\downarrow$ and $\mathcal{T}_\varphi^\updownarrow$ processes where φ will be taken to be an almost surely monotone function. As we will see later, the multidimensional case is very similar to the bidimensional case, so we shall give special attention to the latter.

Given $T \in \mathcal{T}^\updownarrow$, we shall always fix an absolutely continuous T-invariant probability measure and denote it by μ_T. Now let $\varphi \in \mathcal{L}^1(\mu_T)$ be an almost surely increasing function and consider $\{X_t\}_{t\in\mathbb{N}}$ the $\mathcal{T}^\updownarrow$ process associated to T. For all $t \in \mathbb{N}$, let $F_t(\cdot)$ be the distribution function of X_t. By definition, for all $x \in I$

$$F_0(x) := \mathbb{P}(U_0 \leq x) = \mu_T\big([0, x]\big).$$

Observe that the T-invariance of μ_T, implies that, for any $t \in \mathbb{N}$, $t > 0$, and $x \in I$,

$$F_t(x) := \mathbb{P}\big(T^t(U_0) \leq x\big) = \mu_T\left((T^t)^{-1}\big([0, x]\big)\right) = \mu_T\big([0, x]\big) = F_0(x). \tag{3.1}$$

Also, $\mu_T \ll \lambda$ implies that μ_T is non-atomic and since $T \in C^{1+\alpha}(I)$ implies the existence of a continuous positive density for μ_T, F_t is continuous, increasing and its inverse is well defined.

Remark 3.1. Notice that, if $T \in \mathcal{T}^\updownarrow$ has s nodes, for $t \geq 0$ T^t will have s^t nodes and the restriction of T^t to each of its nodes (say $\{I_k\}_{k=1}^{s^t}$) is a one-to-one function, so that on each I_k its inverse is locally well defined. Now, we can conveniently define the inverse of T^t at $y \in (0, 1)$ ($y \in \{0, 1\}$ is trivial) as a piecewise function by setting $\mathcal{T}_{t,k}(y) := (T^t)^{-1}|_{I_k}(y)$, so that

$$(T^t)^{-1}(y) = \big((T^t)^{-1}|_{I_1}(y), \cdots, (T^t)^{-1}|_{I_{s^t}}(y)\big) = \big(\mathcal{T}_{t,1}(y), \cdots, \mathcal{T}_{t,s^t}(y)\big).$$

This is just a simple way of writing the inverse image of the singleton $\{y\}$ by T^t. With this in mind, let $y \in (0, 1)$ ($y \in \{0, 1\}$ is trivial), $t > 0$, X be a random variable taking values in I and $\{a_{t,k}\}_{k=0}^{s^t}$ be the net associated to the nodes of T^t. For $T \in \mathcal{T}^\uparrow \bigcup \mathcal{T}^\downarrow$, the solution of the inequality $T^t(X) \leq y$ in X can be written as $X \in A_{t,1}(y) \bigcup \cdots \bigcup A_{t,s^t}(y)$, where

$$A_{t,k}(y) := \begin{cases} [a_{t,k-1}, \mathcal{T}_{t,k}(y)], & \text{if} \quad T \in \mathcal{T}^\uparrow, \\ [\mathcal{T}_{t,k}(y), a_{t,k}], & \text{if} \quad T \in \mathcal{T}^\downarrow, \end{cases} \tag{3.2}$$

will be a proper closed subinterval of $[a_{t,k-1}, a_{t,k}]$, for each $k = 1, \cdots, s^t$. Notice that $A_{t,k}(y)$ is just the inverse image of $[0, y]$ by the transformation T^t restricted to its k-th node, that is, $A_{t,k}(y) = (T^t)^{-1}\big([0, y]\big) \bigcap I_k$.

The next result will be used several times during the work.

Lemma 3.1. *Let X be a random variable taking values in I and let T belong to either $\mathcal{T}^{\uparrow}$ or $\mathcal{T}^{\downarrow}$ and suppose that T has s nodes. Then, for any $t \in \mathbb{N}$ and $x \in I$,*

$$\mathbb{P}\big(T^t(X) \leq x\big) = \mathbb{P}\big(X \in \textstyle\bigcup_{k=1}^{s^t} A_{t,k}(x)\big) = \displaystyle\sum_{k=1}^{s^t} \mathbb{P}\big(X \in A_{t,k}(x)\big),$$

where the $A_{t,k}$'s are given by (3.2)*.*

Proof: The result follows from Remark 3.1 and from the fact that the $A_{t,k}$'s are (pairwise) disjoint. ■

The case where $T \in \mathcal{T}^{\updownarrow} \setminus (\mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow})$ is treated in the next lemma. As one could expect, it will be a mix of the results in Lemma 3.1.

Lemma 3.2. *Let X be a random variable taking values in I and distributed according to a non-atomic distribution. Let T belong to $\mathcal{T}^{\updownarrow} \setminus (\mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow})$ and suppose that T has s nodes. Then, for any $t \in \mathbb{N}$ and $x \in I$,*

$$\mathbb{P}\big(T^t(X) \leq x\big) = \mathbb{P}\Big(X \in \textstyle\bigcup_{k=1}^{s^t} \big(A_{t,k}^{\uparrow}(x) \bigcup A_{t,k}^{\downarrow}(x)\big)\Big) = \displaystyle\sum_{k=1}^{s^t} \big(\mathbb{P}(X \in A_{t,k}^{\uparrow}(x)) + \mathbb{P}(X \in A_{t,k}^{\downarrow}(x))\big),$$

where

$$A_{t,k}^{\uparrow}(x) := \begin{cases} [a_{t,k-1}, \mathcal{T}_{t,k}(x)], & k \in K_t^{\uparrow}, \\ \emptyset, & \textit{otherwise}, \end{cases} \quad \textit{and} \quad A_{t,k}^{\downarrow}(x) := \begin{cases} [\mathcal{T}_{t,k}(x), a_{t,k}], & k \in K_t^{\downarrow}, \\ \emptyset, & \textit{otherwise}. \end{cases} \tag{3.3}$$

Proof: Let $A_{t,k}^{\uparrow}$ and $A_{t,k}^{\downarrow}$ be as in expression (3.3). Since $T \in \mathcal{T}^{\updownarrow} \setminus (\mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow})$, $K_t^{\uparrow} \subsetneq \{1, \cdots, s^t\}$ and $K_t^{\downarrow} \subsetneq \{1, \cdots, s^t\}$ are non-empty. To prove the first equality, notice that

$$\mathbb{P}\big(T^t(X) \leq x\big) = \mathbb{P}\Big(X \in \big(\textstyle\bigcup_{i \in K_t^{\uparrow}} A_{t,i}^{\uparrow}(x)\big) \bigcup \big(\bigcup_{j \in K_t^{\downarrow}} A_{t,j}^{\downarrow}(x)\big)\Big) = \mathbb{P}\Big(X \in \bigcup_{k=1}^{s^t} \big(A_{t,k}^{\uparrow}(x) \bigcup A_{t,k}^{\downarrow}(x)\big)\Big).$$

As for the second, one can write

$$\begin{aligned} \mathbb{P}\big(T^t(X) \leq x\big) &= \mathbb{P}\Big(X \in \big(\textstyle\bigcup_{i \in K_t^{\uparrow}} A_{t,i}^{\uparrow}(x)\big) \bigcup \big(\bigcup_{j \in K_t^{\downarrow}} A_{t,j}^{\downarrow}(x)\big)\Big) \\ &= \mathbb{P}\Big(X \in \big(\textstyle\bigcup_{i \in K_t^{\uparrow}} A_{t,i}^{\uparrow}(x)\big)\Big) + \mathbb{P}\Big(X \in \big(\bigcup_{j \in K_t^{\downarrow}} A_{t,j}^{\downarrow}(x)\big)\Big) - \\ &\quad - \mathbb{P}\Big(X \in \big(\textstyle\bigcup_{i \in K_t^{\uparrow}} A_{t,i}^{\uparrow}(x)\big) \bigcap \big(\bigcup_{j \in K_t^{\downarrow}} A_{t,j}^{\downarrow}(x)\big)\Big). \end{aligned} \tag{3.4}$$

Now upon noticing that $A_{t,i}^{\uparrow}(x) \bigcap A_{t,j}^{\uparrow}(x) = \emptyset$ and $A_{t,i}^{\downarrow}(x) \bigcap A_{t,j}^{\downarrow}(x) = \emptyset$ whenever $i \neq j$ and by the definition of $A_{t,i}^{\uparrow}$ and $A_{t,i}^{\downarrow}$, it follows that

$$\mathbb{P}\big(X \in \textstyle\bigcup_{i \in K_t^{\uparrow}} A_{t,i}^{\uparrow}(x)\big) = \mathbb{P}\big(X \in \bigcup_{k=1}^{s^t} A_{t,i}^{\uparrow}(x)\big) = \displaystyle\sum_{k=1}^{s^t} \mathbb{P}\big(X \in A_{t,k}^{\uparrow}(x)\big)$$

and similarly $\mathbb{P}\big(X \in \big(\bigcup_{j \in K_t^{\downarrow}} A_{t,j}^{\downarrow}(x)\big)\big) = \sum_{k=1}^{s^t} \mathbb{P}\big(X \in A_{t,k}^{\downarrow}(x)\big)$. At this point, upon substituting these two equalities into (3.4), the lemma will be proved if we show that

$$\mathbb{P}\Big(X \in \big(\textstyle\bigcup_{i \in K_t^{\uparrow}} A_{t,i}^{\uparrow}(x)\big) \bigcap \big(\bigcup_{j \in K_t^{\downarrow}} A_{t,j}^{\downarrow}(x)\big)\Big) = 0.$$

This follows upon observing that

$$\begin{aligned} \mathbb{P}\Big(X \in \big(\textstyle\bigcup_{i \in K_t^{\uparrow}} A_{t,i}^{\uparrow}(x)\big) \bigcap \big(\bigcup_{j \in K_t^{\downarrow}} A_{t,j}^{\downarrow}(x)\big)\Big) &= \mathbb{P}\Big(X \in \textstyle\bigcup_{k=1}^{s^t} \big(A_{t,k}^{\uparrow}(x) \bigcap A_{t,k}^{\downarrow}(x)\big)\Big) \\ &\leq \mathbb{P}\Big(X \in \big(\textstyle\bigcup_{k=0}^{s^t} \{a_{t,k}\}\big) \bigcup \big(\bigcup_{k=1}^{s^t} \{\mathcal{T}_{t,k}(x)\}\big)\Big) = 0, \end{aligned}$$

since, by assumption, X has a non-atomic distribution. This completes the proof. ■

Remark 3.2. Lemma 3.2 is the analogous of Lemma 3.1 in the case where $T\in\mathcal{T}^{\updownarrow}\backslash(\mathcal{T}^{\uparrow}\bigcup\mathcal{T}^{\downarrow})$, but with an extra condition: we had to impose the condition of non-atomicity in the distribution of X. The non-atomicity condition is necessary because the intersection in (3.4) may be non-empty, in which case it will contain only a finite number of isolated points. However, in our work we will always assume that the underline distribution is absolutely continuous, so the extra imposed condition has no impact in the development of the work.

Proposition 3.1. *Let $T\in\mathcal{T}^{\updownarrow}$, μ_T be a T-invariant probability measure and let U_0 be distributed as μ_T. Then, for any $t,h\in\mathbb{N}$, $h\neq 0$, $\left(T^t(U_0),T^{t+h}(U_0)\right)\overset{d}{=}\left(U_0,T^h(U_0)\right)$ and $C_{T^t(U_0),T^{t+h}(U_0)}=C_{U_0,T^h(U_0)}$.*

Proof: The first result follows from the fact that, for $T\in\mathcal{T}^{\updownarrow}$ and $\varphi\in\mathcal{L}^1(\mu_T)$, the $\mathcal{T}^{\updownarrow}_{\varphi}$ process associated is stationary. In particular, the result holds for φ taken to be the identity map. The second result is a consequence of the process' stationarity and of Sklar's theorem. ■

As for the copulas related to $\mathcal{T}^{\updownarrow}_{\varphi}$ processes, for φ almost surely increasing, the following result holds.

Corollary 3.1. *Let $T\in\mathcal{T}^{\updownarrow}$, μ_T be a T-invariant probability measure and $\varphi\in\mathcal{L}^1(\mu_T)$ be an almost everywhere increasing function. Let $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}^{\updownarrow}_{\varphi}$ process. Then, for any $t,h\in\mathbb{N}$, $h\neq 0$,*

$$C_{X_t,X_{t+h}}(u,v)=C_{X_0,X_h}(u,v)=C_{U_0,T^h(U_0)}(u,v),$$

everywhere in I^2.

Proof: Direct consequence of Proposition 3.1 and Theorem 2.1. ■

Remark 3.3. Notice that the result in Corollary 3.1 actually holds in a much more general context when combined with Proposition 3.1, but for our purposes, we only need it as stated.

Now we turn our attention to determining the copula associated to any pair (X_p,X_q) of random variables, for any $p,q\in\mathbb{N}$, obtained from a $\mathcal{T}^{\updownarrow}_{\varphi}$ process with φ increasing almost everywhere. Let $T\in\mathcal{T}^{\updownarrow}_{\varphi}$ and assume that T has $s\geq 1$ nodes. Let μ_T be a T-invariant probability measure and F_0 be as before. In order to simplify the notation, let us define the functions $\mathscr{F}_{h,k}:I\to[F_0(a_{h,k-1}),F_0(a_{h,k})]$ by

$$\mathscr{F}_{h,k}(x):=F_0\Big(\mathcal{T}_{h,k}\big(F_0^{-1}(x)\big)\Big), \tag{3.5}$$

for $h>0$ and $k\in\{1,\cdots,s^h\}$. For a given set S, we also define $\delta_S(u)$ as being 1, if $u\in S$, and 0 otherwise. We start with the case $T\in\mathcal{T}^{\updownarrow}\backslash(\mathcal{T}^{\uparrow}\bigcup\mathcal{T}^{\downarrow})$. We shall denote the copula related to a given $\mathcal{T}^{\updownarrow}_{\varphi}$ process $\{X_t\}_{t\in\mathbb{N}}$ by $C^{\uparrow}_{X_t,X_{t+h}}$, $C^{\downarrow}_{X_t,X_{t+h}}$ and $C^{\updownarrow}_{X_t,X_{t+h}}$ when T belongs to $\mathcal{T}^{\uparrow}$, $\mathcal{T}^{\downarrow}$ and $\mathcal{T}^{\updownarrow}\backslash(\mathcal{T}^{\uparrow}\bigcup\mathcal{T}^{\downarrow})$, respectively.

Proposition 3.2. *Let $T\in\mathcal{T}^{\updownarrow}\backslash(\mathcal{T}^{\uparrow}\bigcup\mathcal{T}^{\downarrow})$, μ_T be a T-invariant probability measure and $\varphi\in\mathcal{L}^1(\mu_T)$ be an almost everywhere increasing function. Let $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}^{\updownarrow}_{\varphi}$ process. If we let $\{a_{h,k}\}_{k=0}^{s^h}$ be the net associated to the nodes of T^h, then*

$$\begin{aligned} C^{\updownarrow}_{X_t,X_{t+h}}(u,v) \;=\; & \sum_{k\in n_0^{\uparrow}}\big[\mathscr{F}_{h,k}(v)-F_0(a_{h,k-1})\big]+\big[\min\{u,\mathscr{F}_{h,n_0}(v)\}-F_0(a_{h,n_0-1})\big]\delta_{K_h^{\uparrow}}(n_0)+\\ & +\sum_{k\in n_0^{\downarrow}}\big[F_0(a_{h,k})-\mathscr{F}_{h,k}(v)\big]+\max\{0,u-\mathscr{F}_{h,n_0}(v)\}\delta_{K_h^{\downarrow}}(n_0), \end{aligned} \tag{3.6}$$

where $n_0:=n_0(u;h)=\big\{k:u\in\big[F_0(a_{h,k-1}),F_0(a_{h,k})\big)\big\}$, and, with $K_h^{\uparrow}$ and $K_h^{\downarrow}$ as in (2.1),

$$n_0^{\uparrow}:=\{1,\cdots,n_0-1\}\bigcap K_h^{\uparrow}\quad\text{and}\quad n_0^{\downarrow}:=\{1,\cdots,n_0-1\}\bigcap K_h^{\downarrow}. \tag{3.7}$$

Proof: We start by determining the copula associated to the pair $\big(U_0, T^h(U_0)\big)$. Let $T \in \mathcal{T}^{\updownarrow}\backslash(\mathcal{T}^{\uparrow}\bigcup\mathcal{T}^{\downarrow})$, μ_T be a T-invariant probability measure and $\varphi \in \mathcal{L}^1(\mu_T)$ be an almost everywhere increasing function. Also let $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}_\varphi^{\updownarrow}$ process and $H_{0,h}(\cdot,\cdot)$ denote the distribution function of the pair $\big(U_0, T^h(U_0)\big)$. Notice that, by Lemma 3.2, we have

$$\begin{aligned}
H_{0,h}(x,y) &= \mathbb{P}(U_0 \le x, T^h(U_0) \le y) = \mathbb{P}\big(U_0 \le x, U_0 \in \textstyle\bigcup_{k=1}^{s^h} A_{h,k}(y)\big)\\
&= \mathbb{P}\big(U_0 \in [0,x]\textstyle\bigcap\bigcup_{k=1}^{s^h} A_{h,k}(y)\big).\\
&= \sum_{k=1}^{s^h}\Big[\mathbb{P}\big(U_0 \in A^{\uparrow}_{h,k}(y)\textstyle\bigcap[0,x]\big) + \mathbb{P}\big(U_0 \in A^{\downarrow}_{h,k}(y)\textstyle\bigcap[0,x]\big)\Big],
\end{aligned}$$

where $A^{\uparrow}_{h,k}$ and $A^{\downarrow}_{h,k}$ are given in (3.3). Set $n_1 := n_1(x;h) = \big\{k : x \in [a_{h,k-1}, a_{h,k})\big\}$ and let

$$n_1^{\uparrow} := \{1,\cdots,n_1-1\}\textstyle\bigcap K_h^{\uparrow} \quad \text{and} \quad n_1^{\downarrow} := \{1,\cdots,n_1-1\}\textstyle\bigcap K_h^{\downarrow},$$

with $K_h^{\uparrow}$ and $K_h^{\downarrow}$ as in (2.1). Notice that $n_1^{\uparrow}\bigcap n_1^{\downarrow} = \emptyset$ and $n_1^{\uparrow}\bigcup n_1^{\downarrow} = \{1,\cdots,n_1-1\}$. It follows that

$$\begin{aligned}
H_{0,h}(x,y) &= \sum_{k=1}^{s^h}\Big[\mathbb{P}\big(U_0 \in A^{\uparrow}_{h,k}(y)\textstyle\bigcap[0,x]\big) + \mathbb{P}\big(U_0 \in A^{\downarrow}_{h,k}(y)\textstyle\bigcap[0,x]\big)\Big]\\
&= \sum_{k\in n_1^{\uparrow}} \mathbb{P}\big(U_0 \in A^{\uparrow}_{h,k}(y)\textstyle\bigcap[0,x]\big) + \mathbb{P}\big(U_0 \in \underbrace{A^{\uparrow}_{h,n_1}(y)\textstyle\bigcap[a_{h,n_1},x]}_{=:Q^{\uparrow}}\big)\delta_{K_h^{\uparrow}}(n_1) +\\
&\quad + \sum_{k\in n_1^{\downarrow}} \mathbb{P}\big(U_0 \in A^{\downarrow}_{h,k}(y)\textstyle\bigcap[0,x]\big) + \mathbb{P}\big(U_0 \in \underbrace{A^{\downarrow}_{h,n_1}(y)\textstyle\bigcap[a_{h,n_1},x]}_{=:Q^{\downarrow}}\big)\delta_{K_h^{\downarrow}}(n_1)\\
&= \sum_{k\in n_1^{\uparrow}} \mu_T\big(\big[a_{h,k-1}, \mathcal{T}_{h,k}(y)\big]\big) + \mu_T(Q^{\uparrow})\delta_{K_h^{\uparrow}}(n_1) +\\
&\qquad\qquad + \sum_{k\in n_1^{\downarrow}} \mu_T\big(\big[\mathcal{T}_{h,k}(y), a_{h,k-1}\big]\big) + \mu_T(Q^{\downarrow})\delta_{K_h^{\downarrow}}(n_1),
\end{aligned}$$

where

$$Q^{\uparrow} = \big[a_{h,n_1-1}, \min\{x, \mathcal{T}_{h,n_1}(y)\}\big] \quad \text{and} \quad Q^{\downarrow} = \begin{cases} \emptyset, & \text{if} \quad x < \mathcal{T}_{h,n_1}(y),\\ \big[\mathcal{T}_{h,n_1}(y), x\big], & \text{if} \quad x \ge \mathcal{T}_{h,n_1}(y),\end{cases}$$

so that

$$\begin{aligned}
H_{0,h}(x,y) = &\sum_{k\in n_1^{\uparrow}}\big[F_0\big(\mathcal{T}_{h,k}(y)\big) - F_0(a_{h,k-1})\big] + \sum_{k\in n_1^{\downarrow}}\big[F_0(a_{h,k}) - F_0\big(\mathcal{T}_{h,k}(y)\big)\big] +\\
&+\big[F_0\big(\min\{x,\mathcal{T}_{h,n_1}(y)\}\big) - F_0(a_{h,n_1-1})\big]\delta_{K_h^{\uparrow}}(n_1) + \max\big\{0, F_0(x) - F_0(\mathcal{T}_{h,n_1}(y))\big\}\delta_{K_h^{\downarrow}}(n_1).
\end{aligned}$$

Upon noticing that $F_0\big(\min\{x,\mathcal{T}_{h,n_1}(y)\}\big) = \min\big\{F_0(x), F_0\big(\mathcal{T}_{h,n_1}(y)\big)\big\}$, by Sklar's Theorem, it follows that

$$\begin{aligned}
C^{\updownarrow}_{U_0,T^h(U_0)}(u,v) &= H_{0,h}\big(F_0^{-1}(u), F_h^{-1}(v)\big) = H_{0,h}\big(F_0^{-1}(u), F_0^{-1}(v)\big)\\
&= \sum_{k\in n_0^{\uparrow}}\big[\mathscr{F}_{h,k}(v) - F_0(a_{h,k-1})\big] + \big[\min\{u, \mathscr{F}_{h,n_0}(v)\} - F_0(a_{h,n_0-1})\big]\delta_{K_h^{\uparrow}}(n_0) +\\
&\qquad + \sum_{k\in n_0^{\downarrow}}\big[F_0(a_{h,k}) - \mathscr{F}_{h,k}(v)\big] + \max\big\{0, u - \mathscr{F}_{h,n_0}(v)\big\}\delta_{K_h^{\downarrow}}(n_0),
\end{aligned}$$

where $n_0 := n_1\big(F_0^{-1}(u);h\big)$ and $n_0^{\uparrow}$ and $n_0^{\downarrow}$ are given by (3.7). Now by Proposition 3.1 and Corollary 3.1, $C^{\updownarrow}_{X_t,X_{t+h}} = C^{\updownarrow}_{U_0,T^h(U_0)}$ and the desired result follows. ■

Remark 3.4. Let $\{a_k\}_{k\in\mathbb{N}}$ be an arbitrary sequence of real numbers and let $D \subseteq \mathbb{N}$ be a set of indexes. As usual, if $D = \emptyset$, we shall set $\sum_{k\in D} a_k = 0$.

Next we shall derive the copula associated to any pair (X_p, X_q), for any $p, q \in \mathbb{N}$, of random variables obtained from a $\mathcal{T}_\varphi^\uparrow$ or $\mathcal{T}_\varphi^\downarrow$ process with φ increasing almost everywhere. The results follow from a specialization of Proposition 3.2.

Proposition 3.3. *Let $T \in \mathcal{T}^\uparrow \bigcup \mathcal{T}^\downarrow$, μ_T be a T-invariant probability measure and $\varphi \in \mathcal{L}^1(\mu_T)$ be an almost everywhere increasing function. Let $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}_\varphi^\uparrow$ or $\mathcal{T}_\varphi^\downarrow$ process. If we let $\{a_{h,k}\}_{k=0}^{s^h}$ be the net associated to the nodes of T^h and $n_0 := n_0(u;h) = \big\{k : u \in \big[F_0(a_{h,k-1}), F_0(a_{h,k})\big)\big\}$, then the following is true:*

(i). *If $T \in \mathcal{T}^\uparrow$, then, for any $t, h \in \mathbb{N}$, $h \neq 0$ and $(u, v) \in I^2$,*

$$C^\uparrow_{X_t,X_{t+h}}(u,v) = \left(\sum_{k=1}^{n_0-1} \mathscr{F}_{h,k}(v) - F_0(a_{h,k-1})\right)\delta_{\mathbb{N}^*}(n_0-1) + \\ + \min\big\{u, \mathscr{F}_{h,n_0}(v)\big\} - F_0(a_{h,n_0-1}), \tag{3.8}$$

where $\mathbb{N}^* := \mathbb{N}\backslash\{0\}$.

(ii). *If $T \in \mathcal{T}^\downarrow$ instead, then, for any $t, h \in \mathbb{N}$, $h \neq 0$ and $(u, v) \in I^2$,*

$$C^\downarrow_{X_t,X_{t+h}}(u,v) = \left(\sum_{k=1}^{n_0-1} F_0(a_{h,k}) - \mathscr{F}_{h,k}(v)\right)\delta_{\mathbb{N}^*}(n_0-1) + \max\big\{0, u - \mathscr{F}_{h,n_0}(v)\big\}. \tag{3.9}$$

Proof: If $T \in \mathcal{T}^\uparrow$, (i) follows from Proposition 3.2 by noticing that $K_h^\downarrow = \emptyset$ while if $T \in \mathcal{T}^\downarrow$, the opposite happens, namely, $K_h^\uparrow = \emptyset$ which implies (ii). ∎

Remark 3.5. For computational purposes, it can be advantageous to write copulas (3.6), (3.8) and (3.9) explicitly as a function of μ_T. In this case, (3.6) becomes

$$C^\updownarrow_{X_t,X_{t+h}}(u,v) = \sum_{k\in n_0^\uparrow} \mu_T\Big(\big[a_{h,k-1}, \mathcal{T}_{h,k}\big(F_0^{-1}(v)\big)\big]\Big) + \\ + \mu_T\Big(\big[a_{h,n_0-1}, \min\big\{F_0^{-1}(u), \mathcal{T}_{h,n_0}\big(F_0^{-1}(v)\big)\big\}\big]\Big)\delta_{K_h^\uparrow}(n_0) + \\ + \sum_{k\in n_0^\downarrow} \mu_T\Big(\big[\mathcal{T}_{h,k}\big(F_0^{-1}(v)\big), a_{h,k}\big]\Big) + \mu_T\Big(\big[\mathcal{T}_{h,n_0}\big(F_0^{-1}(v)\big), F_0^{-1}(u)\big]^+\Big)\delta_{K_h^\downarrow}(n_0), \tag{3.10}$$

where $[a, b]^+$ equals $[a, b]$, if $b \geq a$, and $\emptyset$ otherwise. In their turn, copulas (3.8) and (3.9) become respectively

$$C^\uparrow_{X_t,X_{t+h}}(u,v) = \sum_{k=1}^{n_0-1} \mu_T\Big(\big[a_{h,k-1}, \mathcal{T}_{h,k}\big(F_0^{-1}(v)\big)\big]\Big)\delta_{\mathbb{N}^*}(n_0-1) + \\ + \mu_T\Big(\big[a_{h,n_0-1}, \min\big\{F_0^{-1}(u), \mathcal{T}_{h,n_0}\big(F_0^{-1}(v)\big)\big\}\big]\Big),$$

and

$$C^\downarrow_{X_t,X_{t+h}}(u,v) = \sum_{k=1}^{n_0-1} \mu_T\Big(\big[\mathcal{T}_{h,k}\big(F_0^{-1}(v)\big), a_{h,k}\big]\Big)\delta_{\mathbb{N}^*}(n_0-1) + \mu_T\Big(\big[\mathcal{T}_{h,n_0}\big(F_0^{-1}(v)\big), F_0^{-1}(u)\big]^+\Big).$$

In the next lemma we show that the relation $C_{X_t,X_{t+h}} = C_{X_0,X_h}$, valid when φ is increasing, still holds in the decreasing case.

Lemma 3.3. *Let $T \in \mathcal{T}^{\updownarrow}$ and let $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}_{\varphi}^{\updownarrow}$ process for $\varphi \in \mathcal{L}^1(\mu_T)$ an almost surely decreasing function. Then, the relation*

$$C_{X_t,X_{t+h}}(u,v) = u+v-1+C_{U_0,T^h(U_0)}(1-u,1-v) = C_{X_0,X_h}(u,v), \tag{3.11}$$

holds for all $(u,v) \in I^2$.

Proof: In Proposition 3.1, we have shown that $C_{T^t(U_0),T^{t+h}(U_0)} = C_{U_0,T^h(U_0)}$ (independently of φ). Now, since the inverse of an almost everywhere decreasing function is still decreasing almost everywhere, upon applying Theorem 2.1, it follows that

$$C_{X_t,\,X_{t+h}}(u,v) = C_{\varphi^{-1}(T^t(U_0)),\,\varphi^{-1}(T^{t+h}(U_0))}(u,v) = u+v-1+C_{T^t(U_0),\,T^{t+h}(U_0)}(1-u,1-v),$$

which proves the first equality. As for the second, it suffices to notice that

$$C_{X_0,X_h}(u,v) = C_{\varphi(U_0),\varphi(T^h(U_0))}(u,v) = u+v-1+C_{U_0,\,T^h(U_0)}(1-u,1-v),$$

everywhere in I^2, by Theorem 2.1, and the result follows by Proposition 3.1. ■

In the next propositions we present the copulas related to $\mathcal{T}_{\varphi}^{\updownarrow}$ processes in the case where $\varphi \in \mathcal{L}^1(\mu_t)$ is an almost everywhere decreasing function. In this case, we use the same notation for the copulas as before, but we add an asterisk in order to emphasize the difference on φ. We start by considering $T \in \mathcal{T}^{\updownarrow} \setminus (\mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow})$.

Proposition 3.4. *Let $T \in \mathcal{T}^{\updownarrow} \setminus (\mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow})$ be a transformation with $s > 1$ nodes, μ_T be a T-invariant probability measure and $\varphi \in \mathcal{L}^1(\mu_T)$ be an almost everywhere decreasing function. Let $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}_{\varphi}^{\updownarrow}$ process. If we let $\{a_{h,k}\}_{k=0}^{s^h}$ be the net associated to the nodes of T^h, then*

$$\begin{aligned} C^{\updownarrow *}_{X_t,X_{t+h}}(u,v) \;=\;& u+v-1+\sum_{k\in n_0^{\uparrow *}}\big[\mathscr{F}_{h,k}(1-v)-F_0(a_{h,k-1})\big]+\sum_{k\in n_0^{\downarrow *}}\big[F_0(a_{h,k})-\mathscr{F}_{h,k}(1-v)\big]+\\ &+\big[\min\big\{1-u,\mathscr{F}_{h,n_0^*}(1-v)\big\}-F_0(a_{h,n_0^*-1})\big]\delta_{K_h^{\uparrow}}(n_0^*)+\\ &+\max\big\{0,1-u-\mathscr{F}_{h,n_0^*}(1-v)\big\}\delta_{K_h^{\downarrow}}(n_0^*), \end{aligned} \tag{3.12}$$

where $n_0^* := n_0^*(u;h) = \big\{k : u \in \big[1-F_0(a_{h,k}),1-F_0(a_{h,k-1})\big)\big\}$,

$$n_0^{\uparrow *} := \{1,\cdots,n_0^*-1\}\bigcap K_h^{\uparrow} \quad \textit{and} \quad n_0^{\downarrow *} := \{1,\cdots,n_0^*-1\}\bigcap K_h^{\downarrow}.$$

Proof: By Lemma 3.3, we have

$$C^{\updownarrow *}_{X_t,X_{t+h}}(u,v) = u+v-1+C^{\updownarrow}_{U_0,T^h(U_0)}(1-u,1-v) \tag{3.13}$$

so that the result follows by Proposition 3.2 (with φ as the identity map) upon substituting (3.6) into (3.13). ■

Proposition 3.5. *Let $T \in \mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow}$, μ_T be a T-invariant probability measure and $\varphi \in \mathcal{L}^1(\mu_T)$ be an almost everywhere decreasing function. Let $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}_{\varphi}^{\uparrow}$ or $\mathcal{T}_{\varphi}^{\downarrow}$ process. If we let $\{a_{h,k}\}_{k=0}^{s^h}$ be the net associated to the nodes of T^h and $n_0^* := n_0^*(u;h) = \big\{k : u \in \big[1-F_0(a_{h,k}),1-F_0(a_{h,k-1})\big)\big\}$, then the following is true:*

(i). *If $T \in \mathcal{T}^{\uparrow}$, then, for any $t,h \in \mathbb{N}$, $h \neq 0$ and $(u,v) \in I^2$,*

$$\begin{aligned} C^{\uparrow *}_{X_t,X_{t+h}}(u,v) \;=\;& u+v-1+\left(\sum_{k=1}^{n_0^*-1}\mathscr{F}_{h,k}(1-v)-F_0(a_{h,k-1})\right)\delta_{\mathbb{N}^*}(n_0^*-1)+\\ &+\min\big\{1-u,\mathscr{F}_{h,n_0^*}(1-v)\big\}-F_0(a_{h,n_0^*-1}). \end{aligned} \tag{3.14}$$

(ii). *If $T \in \mathcal{T}^{\downarrow}$, then, for any $t, h \in \mathbb{N}$, $h \neq 0$ and $(u,v) \in I^2$,*

$$C^{\downarrow *}_{X_t,X_{t+h}}(u,v) = u + v - 1 + \left(\sum_{k=1}^{n_0^*-1} F_0(a_{h,k}) - \mathscr{F}_{h,k}(1-v)\right) \delta_{\mathbb{N}^*}(n_0^* - 1) + \\ + \max\left\{0, 1 - u - \mathscr{F}_{h,n_0^*}(1-v)\right\}. \tag{3.15}$$

Proof: Immediate from Proposition 3.4 and (3.13). ∎

A question that naturally arises is what are the support of the copulas just derived? To answer that, the following notation will be useful. Let $T \in \mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow}$ and F_0 be the distribution associated to a T-invariant probability measure. Assume that T has $s > 1$ nodes and let h be a positive integer. We define, for all $k \in \{1, \cdots, s^h\}$, functions $\ell^{\uparrow}_{h,k}, \ell^{\downarrow}_{h,k} : \left[F_0(a_{h,k-1}), F_0(a_{h,k})\right] \to I$ by setting

$$\ell^{\uparrow}_{h,k}(x) := \frac{x - F_0(a_{h,k-1})}{F_0(a_{h,k}) - F_0(a_{h,k-1})} \quad \text{and} \quad \ell^{\downarrow}_{h,k}(x) := \frac{F_0(a_{h,k}) - x}{F_0(a_{h,k}) - F_0(a_{h,k-1})}$$

and $\ell^{\uparrow *}_{h,k}, \ell^{\downarrow *}_{h,k} : \left[1 - F_0(a_{h,k}), 1 - F_0(a_{h,k-1})\right] \to I$ by setting

$$\ell^{\uparrow *}_{h,k}(x) := \frac{x + F_0(a_{h,k-1}) - 1}{F_0(a_{h,k}) - F_0(a_{h,k-1})} \quad \text{and} \quad \ell^{\downarrow *}_{h,k}(x) := \frac{1 - F_0(a_{h,k}) - x}{F_0(a_{h,k}) - F_0(a_{h,k-1})}.$$

Observe that $\ell^{\uparrow}_{h,k}$ is just the linear function joining $(F_0(a_{h,k-1}), 0)$ and $(F_0(a_{h,k}), 1)$, while $\ell^{\downarrow}_{h,k}$ joins $(F_0(a_{h,k-1}), 1)$ and $(F_0(a_{h,k}), 0)$. In the next proposition we provide a characterization for the support of the copulas derived so far.

Proposition 3.6. *Let $T \in \mathcal{T}^{\updownarrow}$ and μ_T be a T-invariant probability measure. For $\varphi_1 \in \mathcal{L}^1(\mu_T)$ an almost everywhere increasing function and $\varphi_2 \in \mathcal{L}^1(\mu_T)$ an almost everywhere decreasing function, let $\{X_t\}_{t\in\mathbb{N}}$ and $\{Y_t\}_{t\in\mathbb{N}}$ denote respectively the associated $\mathcal{T}^{\updownarrow}_{\varphi_1}$ and $\mathcal{T}^{\updownarrow}_{\varphi_2}$ process. Also suppose that T has $s \geq 1$ nodes. Then, for any $t, h \in \mathbb{N}$ and $h > 0$,*

$$\mathrm{supp}\{C^{\updownarrow}_{X_t,X_{t+h}}\} = \Big(\bigcup_{k\in K^{\uparrow}} \big\{\big(u, \ell^{\uparrow}_{h,k}(u)\big) : u \in R_{h,k}\big\}\Big) \bigcup \Big(\bigcup_{k\in K^{\downarrow}} \big\{\big(u, \ell^{\downarrow}_{h,k}(u)\big) : u \in R_{h,k}\big\}\Big),$$

and

$$\mathrm{supp}\{C^{\updownarrow}_{Y_t,Y_{t+h}}\} = \Big(\bigcup_{k\in K^{\uparrow}} \big\{\big(u, \ell^{\downarrow *}_{h,k}(u)\big) : u \in R^*_{h,k}\big\}\Big) \bigcup \Big(\bigcup_{k\in K^{\downarrow}} \big\{\big(u, \ell^{\uparrow *}_{h,k}(u)\big) : u \in R^*_{h,k}\big\}\Big),$$

where $R_{h,k} := \left[F_0(a_{h,k-1}), F_0(a_{h,k})\right]$ and $R^*_{h,k} := \left[1 - F_0(a_{h,k}), 1 - F_0(a_{h,k-1})\right]$.

Proof: Let $R = [u_1, u_2] \times [v_1, v_2]$ be a rectangle in I^2. First assume that $T \in \mathcal{T}^{\uparrow}_{\varphi_1}$ and let $V_{C^{\uparrow}_{\boldsymbol{X}}}(R)$ denote the $C_{X_t,X_{t+h}}$-volume of R. Let $k \in \{1, \cdots, s^h\}$ be fixed and since for any copula C, the C-volume is a (doubly stochastic) measure, we can assume without loss of generality that $u_i \in R_{h,k}$, for $i = 1, 2$, so that $n_0 = k$ for all terms in the expression of $V_{C^{\uparrow}_{\boldsymbol{X}}}(R)$. Hence the summands and constants on the copula cancel out so that we have

$$V_{C^{\uparrow}_{\boldsymbol{X}}}(R) = \min\left\{u_1, \mathscr{F}_{h,k}(v_1)\right\} + \min\left\{u_2, \mathscr{F}_{h,k}(v_2)\right\} - \min\left\{u_1, \mathscr{F}_{h,k}(v_2)\right\} - \min\left\{u_2, \mathscr{F}_{h,k}(v_1)\right\} \\ = V_M\big([u_1, u_2] \times \left[\mathscr{F}_{h,k}(v_1), \mathscr{F}_{h,k}(v_2)\right]\big),$$

where $M(u,v) = \min\{u,v\}$ is the Frechèt upper bound copula, whose support is the main diagonal in I^2. Since $[u_1, u_2] \times \left[\mathscr{F}_{h,k}(v_1), \mathscr{F}_{h,k}(v_2)\right] \subseteq R^2_{h,k}$, it follows that $V_{C^{\uparrow}_{\boldsymbol{X}}}(R) > 0$ if, and only if, $R \bigcap \big\{\big(u, \ell^{\uparrow}_{h,k}(u)\big) : u \in R_{h,k}\big\} \neq \emptyset$.

Now assume that $T \in \mathcal{T}^{\downarrow}_{\varphi_1}$ and, considering a rectangle R as before and $u_i \in R_{h,k}$, let $V_{C^{\downarrow}_{\boldsymbol{X}}}(R)$ denote its $C_{X_t,X_{t+h}}$-volume. Again the summands and constants on the copula cancel out and we have

$$V_{C^{\downarrow}_{\boldsymbol{X}}}(R) = \max\left\{0, u_1 - \mathscr{F}_{h,k}(v_1)\right\} + \max\left\{0, u_2 - \mathscr{F}_{h,k}(v_2)\right\} - \max\left\{0, u_1 - \mathscr{F}_{h,k}(v_2)\right\} -$$

$$\begin{aligned}&\quad - \max\{0, u_2 - \mathscr{F}_{h,k}(v_1)\}\\ &= V_W\big([u_1,u_2]\times\big[1-\mathscr{F}_{h,k}(v_1), 1-\mathscr{F}_{h,k}(v_2)\big]\big),\end{aligned}$$

where $W(u,v) = \max\{0, u+v-1\}$ is the Frechèt lower bound copula, whose support is the secondary diagonal in I^2. Since $[u_1,u_2]\times\big[1-\mathscr{F}_{h,k}(v_1), 1-\mathscr{F}_{h,k}(v_2)\big] \subseteq R_{h,k}\times R^*_{h,k}$, it follows that $V_{C^{\downarrow}_{\mathbf{X}}}(R) > 0$ if, and only if, $R\bigcap\big\{\big(u,\ell^{\downarrow}_{h,k}(u)\big) : u\in R_{h,k}\big\}\neq\emptyset$.

Now suppose $T\in\mathcal{T}^{\updownarrow}\backslash(\mathcal{T}^{\uparrow}\bigcup\mathcal{T}^{\downarrow})$ and let $V_{C^{\updownarrow}_{\mathbf{X}}}(R)$ denote the $C_{X_t,X_{t+h}}$-volume of R. Let k be fixed as before and let $u_i\in R_{h,k}$, $i=1,2$. We can write

$$V_{C^{\updownarrow}_{\mathbf{X}}}(R) = V_{C^{\uparrow}_{\mathbf{X}}}(R)\delta_{K^{\uparrow}_h}(n_0) + V_{C^{\downarrow}_{\mathbf{X}}}(R)\delta_{K^{\downarrow}_h}(n_0),$$

so that the result follows from the previous cases and by observing that we can write $I = \bigcup_{k=1}^{s^h} R_{h,k}$.

As for $T\in\mathcal{T}^{\updownarrow}_{\varphi_2}$, the result follows similarly as the previous case, by noticing that for a rectangle R as before and $u_i\in R^*_{h,k}$, $i=1,2$,

$$V_{C^{\uparrow}_{\mathbf{Y}}}(R) = V_M\big([1-u_2, 1-u_1]\times\big[\mathscr{F}_{h,k}(1-v_2), \mathscr{F}_{h,k}(1-v_1)\big]\big)$$

and

$$V_{C^{\downarrow}_{\mathbf{Y}}}(R) = V_W\big([1-u_2, 1-u_1]\times\big[1-\mathscr{F}_{h,k}(1-v_2), 1-\mathscr{F}_{h,k}(1-v_1)\big]\big),$$

and by applying standard arguments. This completes the proof. ■

As an immediate consequence, we have

Corollary 3.2. *Let $T\in\mathcal{T}^{\updownarrow}$ and μ_T be a T-invariant probability measure. For $\varphi\in\mathcal{L}^1(\mu_T)$ an almost everywhere monotone function, let $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}^{\updownarrow}_{\varphi}$ process. Then $C^{\updownarrow}_{X_t,X_{t+h}}\}$ is singular with respect to the bidimensional Lebesgue measure.*

Proof: This follows by noticing that $\lambda\big(\mathrm{supp}\{C^{\updownarrow}_{X_t,X_{t+h}}\}\big) = 0$, where λ denotes the Lebesgue measure on I^2. ■

A map $T\in\mathcal{T}^{\updownarrow}$ can have several different absolutely continuous T-invariant probability measures. This implies that, for a single given $T\in\mathcal{T}^{\updownarrow}$, the associated $\mathcal{T}^{\updownarrow}_{\varphi}$ process can have several different copulas associated to it depending on the choice of the T-invariant probability measure. This happens because U_0, which fundamentally defines the probability structure of the process, also depends completely on the choice of the T-invariant probability measure. We also remark that the formulas just presented are all derived from the relation $C_{X_0,T^n(U_0)}(u,v) = \lambda\big([0,u]\bigcap(T^n)^{-1}\big([0,v]\big)\big)$.

4 Multidimensional Case

In this section we shall extend our results from the previous bidimensional set up to a multidimensional one. That is, in this section we are interested in deriving the copulas related to n-dimensional vectors $(X_{t_1},\cdots,X_{t_n})$ coming from a $\mathcal{T}^{\updownarrow}_{\varphi}$ process for φ an almost everywhere monotone function. It turns out that the bidimensional and the multidimensional case have more in common than one could expect and much of the work will be built over the results of last section.

First, let us establish some useful notation. Let $a,b\in\mathbb{N}$ with $a<b$. We shall write $a{:}b := \{a,\cdots,b\}$, $x_{a:b} := (x_a,\cdots,x_b)$ and, for a function f, we shall write $f(x_{a:b}) := \big(f(x_a),\cdots,f(x_b)\big)$. Again we shall denote the distribution of U_0 by F_0. The next proposition will be useful to simplify the proofs of the main results of this section and in establishing notation.

Proposition 4.1. *Let $T \in \mathcal{T}^{\updownarrow} \setminus (\mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow})$ and suppose T has $s \geq 1$ nodes. Denote by $\{a_{t,k}\}_{k=0}^{s^t}$ the net associated to the nodes of T^t. Let $t, h_1, \cdots, h_n \in \mathbb{N}$, $0 < h_1 < \cdots < h_n$, set $\boldsymbol{t} = (t, t+h_1, \cdots, t+h_n)$ and let $H_{\boldsymbol{t}}$ denote the distribution function of $\big(T^t(U_0), T^{t+h_1}(U_0), \cdots, T^{t+h_n}(U_0)\big)$. Then, for all $(x_0, \cdots, x_n) \in I^{n+1}$,*

$$H_{\boldsymbol{t}}(x_0, \cdots, x_n) = \sum_{k \in K_{h_n}^{\uparrow}} \mathbb{P}\big(U_0 \in \widetilde{A}_{h_n,k}^{\uparrow}(x_{1:n}) \bigcap [0, x_0]\big) + \sum_{k \in K_{h_n}^{\downarrow}} \mathbb{P}\big(U_0 \in \widetilde{A}_{h_n,k}^{\downarrow}(x_{1:n}) \bigcap [0, x_0]\big), \tag{4.1}$$

where

$$\widetilde{A}_{h_n,k}^{\uparrow}(x_{1:n}) = \begin{cases} \big[a_{h_n,k-1}, b_{h_n,k}^{\uparrow}(x_{1:n})\big], & \text{if } k \in K_{h_n}^{\uparrow}, \\ \emptyset, & \text{otherwise,} \end{cases} \tag{4.2}$$

and

$$\widetilde{A}_{h_n,k}^{\downarrow}(x_{1:n}) = \begin{cases} \big[b_{h_n,k}^{\downarrow}(x_{1:n}), a_{h_n,k}\big], & \text{if } k \in K_{h_n}^{\downarrow}, \\ \emptyset, & \text{otherwise,} \end{cases} \tag{4.3}$$

with,

$$b_{h_n,k}^{\uparrow}(x_{1:n}) = \min_{i=1,\cdots,n} \big\{c_i^{\uparrow}(x_i; h_n, k)\big\} \quad \text{and} \quad b_{h_n,k}^{\downarrow}(x_{1:n}) = \min_{i=1,\cdots,n} \big\{c_i^{\downarrow}(x_i; h_n, k)\big\} \tag{4.4}$$

where

$$c_i^{\uparrow}(x_i; h_n, k) = \begin{cases} a_{h_n,k-1}, & \text{if } B_i^{\uparrow}(x_i; h_n, k) = \emptyset, \\ B_i^{\uparrow}(x_i; h_n, k), & \text{otherwise,} \end{cases}$$

$$c_i^{\downarrow}(x_i; h_n, k) = \begin{cases} a_{h_n,k}, & \text{if } B_i^{\downarrow}(x_i; h_n, k) = \emptyset, \\ B_i^{\downarrow}(x_i; h_n, k), & \text{otherwise,} \end{cases}$$

$$B_i^{\uparrow}(x_i; h_n, k) = \min_{j=1,\cdots,s^{h_i}} \big\{\mathcal{T}_{h_i,j}(x_i) : \mathcal{T}_{h_i,j}(x_i) > a_{h_n,k-1} \text{ and } a_{h_i,j} < a_{h_n,k}\big\},$$

and

$$B_i^{\downarrow}(x_i; h_n, k) = \max_{j=1,\cdots,s^{h_i}} \big\{\mathcal{T}_{h_i,j}(x_i) : \mathcal{T}_{h_i,j}(x_i) < a_{h_n,k} \text{ and } a_{h_i,j} > a_{h_n,k-1}\big\}.$$

Proof: In view of Theorem 2.1, given $t, h_1, \cdots, h_n \in \mathbb{N}$, $0 < h_1 < \cdots < h_n$ it suffices to prove the result for the vector $\big(T^t(U_0), T^{t+h_1}(U_0), \cdots, T^{t+h_n}(U_0)\big)$. Let $H_{\boldsymbol{t}}$ denote the distribution function of $\big(T^t(U_0), T^{t+h_1}(U_0), \cdots, T^{t+h_n}(U_0)\big)$ and let $x_{0:n} \in (0,1)^{n+1}$. Also, for the sake of simplicity, let $Y_t := T^t(U_0)$. We have

$$\begin{aligned}
H_{\boldsymbol{t}}(x_0, \cdots, x_n) &= \mathbb{P}\big(T^t(U_0) \leq x_0, T^{t+h_1}(U_0) \leq x_1, \cdots, T^{t+h_n}(U_0) \leq x_n\big) \\
&= \mathbb{P}\big(Y_t \leq x_0, T^{h_1}(Y_t) \leq x_1, \cdots, T^{h_n}(Y_t)(U_0) \leq x_n\big) \\
&= \mathbb{P}\Big(Y_t \in [0, x_0], Y_t \in \textstyle\bigcup_{k=1}^{s^{h_1}} \big(A_{h_1,k}^{\uparrow}(x_1) \bigcup A_{h_1,k}^{\downarrow}(x_1)\big), \cdots, Y_t \in \bigcup_{k=1}^{s^{h_n}} \big(A_{h_n,k}^{\uparrow}(x_n) \bigcup A_{h_n,k}^{\downarrow}(x_n)\big)\Big) \\
&= \mathbb{P}\Big(Y_t \in [0, x_0] \textstyle\bigcap_{i=1}^{n} \bigcup_{k=1}^{s^{h_i}} \big(A_{h_i,k}^{\uparrow}(x_i) \bigcup A_{h_i,k}^{\downarrow}(x_i)\big)\Big) \\
&= \mathbb{P}\Big(U_0 \in [0, x_0] \textstyle\bigcap_{i=1}^{n} \bigcup_{k=1}^{s^{h_i}} \big(A_{h_i,k}^{\uparrow}(x_i) \bigcup A_{h_i,k}^{\downarrow}(x_i)\big)\Big) \qquad (4.5) \\
&= \mathbb{P}\Big(U_0 \in [0, x_0] \textstyle\bigcap_{i=1}^{n} \bigcup_{k=1}^{s^{h_i}} A_{h_i,k}^{\uparrow}(x_i)\Big) + \mathbb{P}\Big(U_0 \in [0, x_0] \bigcap_{i=1}^{n} \bigcup_{k=1}^{s^{h_i}} A_{h_i,k}^{\downarrow}(x_i)\Big) - \\
&\quad - \mathbb{P}\Big(U_0 \in [0, x_0] \textstyle\bigcap_{i=1}^{n} \bigcup_{k=1}^{s^{h_i}} \big(A_{h_i,k}^{\uparrow}(x_i) \bigcap A_{h_i,k}^{\downarrow}(x_i)\big)\Big), \qquad (4.6)
\end{aligned}$$

where $A_{h_i,k}^{\uparrow}$ and $A_{h_i,k}^{\downarrow}$ are given in (3.3). The last term in (4.6) is equal to zero since the non-atomicity of the distribution of U_0 implies

$$\begin{aligned}
&\mathbb{P}\Big(U_0 \in [0, x_0] \textstyle\bigcap_{i=1}^{n} \bigcup_{k=1}^{s^{h_i}} \big(A_{h_i,k}^{\uparrow}(x_i) \bigcap A_{h_i,k}^{\downarrow}(x_i)\big)\Big) \leq \\
&\qquad \leq \mathbb{P}\Big(U_0 \in [0, x_0] \textstyle\bigcap \big[\big(\bigcup_{k=0}^{s^{h_n}} \{a_{h_n,k}\}\big) \bigcup \big(\bigcup_{k=1}^{s^{h_n}} \{\mathcal{T}_{h_n,k}(x)\}\big)\big]\Big) \\
&\qquad \leq \mathbb{P}\Big(U_0 \in \big[\big(\textstyle\bigcup_{k=0}^{s^{h_n}} \{a_{h_n,k}\}\big) \bigcup \big(\bigcup_{k=1}^{s^{h_n}} \{\mathcal{T}_{h_n,k}(x)\}\big)\big]\Big) = 0.
\end{aligned}$$

Considering $K^{\uparrow}_{h_n}$ and $K^{\downarrow}_{h_n}$ given by (2.1), let $b^{\uparrow}_{h_n,k}(x_{1:n})$, $b^{\downarrow}_{h_n,k}(x_{1:n})$, $\widetilde{A}^{\uparrow}_{h_n,k}(x_{1:n})$ and $\widetilde{A}^{\downarrow}_{h_n,k}(x_{1:n})$ be as in the enunciate. Notice that $\widetilde{A}^{\uparrow}_{h_n,k}(x_{1:n})$ and $\widetilde{A}^{\downarrow}_{h_n,k}(x_{1:n})$ are both proper closed subsets of $[a_{h_n,k-1}, a_{h_n,k}]$. Each $b^{\uparrow}_{h_n,k}(x_{1:n})$ actually is the smallest $\mathcal{T}_{h_i,j}(x_i)$ that lies on the k-th node of T^{h_n} (which determines the thinner partition among all T^{h_i}), so that $\widetilde{A}^{\uparrow}_{h_n,k}(x_{1:n})$'s are just the intersection of all $A_{h_i,k}(x_i)$'s with the k-th node of T^{h_n}. In opposition, $b^{\downarrow}_{h_n,k}(x_{1:n})$ is the largest $\mathcal{T}_{h_i,j}(x_i)$ that lies on the k-th node of T^{h_n}. Also notice that the $\widetilde{A}^{\uparrow}_{h_n,k}(x_{1:n})$'s are pairwise disjoint and so are the $\widetilde{A}^{\downarrow}_{h_n,k}(x_{1:n})$'s. Now, with this machinery, we can rewrite (4.6) as

$$\begin{aligned} H_{\boldsymbol{t}}(x_0,\cdots,x_n) &= \mathbb{P}\Big(U_0 \in \bigcup\nolimits_{k\in K^{\uparrow}_{h_n}} \widetilde{A}^{\uparrow}_{h_n,k}(x_{1:n}) \bigcap [0,x_0]\Big) + \mathbb{P}\Big(U_0 \in \bigcup\nolimits_{k\in K^{\downarrow}_{h_n}} \widetilde{A}^{\downarrow}_{h_n,k}(x_{1:n}) \bigcap [0,x_0]\Big) \\ &= \sum_{k\in K^{\uparrow}_{h_n}} \mathbb{P}\big(U_0 \in \widetilde{A}^{\uparrow}_{h_n,k}(x_{1:n}) \bigcap [0,x_0]\big) + \sum_{k\in K^{\downarrow}_{h_n}} \mathbb{P}\big(U_0 \in \widetilde{A}^{\downarrow}_{h_n,k}(x_{1:n}) \bigcap [0,x_0]\big), \end{aligned}$$

which is the desired formula. ■

Proposition 4.2. *Let $T \in \mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow}$ and suppose T has $s \geq 1$ nodes and denote by $\{a_{t,k}\}_{k=0}^{s^t}$ the net associated to the nodes of T^t. Let $t, h_1, \cdots, h_n \in \mathbb{N}$, $0 < h_1 < \cdots < h_n$ and $H_{\boldsymbol{t}}$ denote the distribution function of $\big(T^t(U_0), T^{t+h_1}(U_0), \cdots, T^{t+h_n}(U_0)\big)$. Then, for all $(x_0, \cdots, x_n) \in I^{n+1}$,*

$$H_{\boldsymbol{t}}(x_0,\cdots,x_n) = \sum_{k=1}^{s^{h_n}} \mathbb{P}\big(U_0 \in \widetilde{A}_{h_n,k}(x_{1:n}) \bigcap [0,x_0]\big), \tag{4.7}$$

where

$$\widetilde{A}_{h_n,k}(x_{1:n}) = \begin{cases} \big[a_{h_n,k-1}, b^{\uparrow}_{h_n,k}(x_{1:n})\big], & \text{if } T \in \mathcal{T}^{\uparrow}, \\ \big[b^{\downarrow}_{h_n,k}(x_{1:n}), a_{h_n,k}\big], & \text{if } T \in \mathcal{T}^{\downarrow}, \end{cases} \tag{4.8}$$

with $b^{\uparrow}_{h_n,k}(x_{1:n})$ and $b^{\downarrow}_{h_n,k}(x_{1:n})$ given by (4.4).

Proof: With the notation of Proposition 4.1, the result follows by noticing that $\widetilde{A}_{h_n,k}$ is just a combination of (4.2) and (4.3) and that if $T \in \mathcal{T}^{\uparrow}$, then $K^{\downarrow}_{h_n} = \emptyset$, while if $T \in \mathcal{T}^{\uparrow}$, we have $K^{\uparrow}_{h_n} = \emptyset$. ■

Proposition 4.3. *Let $T \in \mathcal{T}^{\updownarrow}$, μ_T be a T-invariant probability measure and let U_0 be distributed as μ_T. Then, for any $t, h_1, \cdots, h_n \in \mathbb{N}$, $0 < h_1 < \cdots < h_n$,*

$$\big(T^t(U_0), T^{t+h_1}(U_0), \cdots, T^{t+h_n}(U_0)\big) \stackrel{d}{=} \big(U_0, T^{h_1}(U_0), \cdots, T^{h_n}(U_0)\big). \tag{4.9}$$

Furthermore, $\big(T^t(U_0), T^{t+h_1}(U_0), \cdots, T^{t+h_n}(U_0)\big)$ and $\big(U_0, T^{h_1}(U_0), \cdots, T^{h_n}(U_0)\big)$ have the same copula.

Proof: Let $H_{\boldsymbol{h}}$ denote the distribution of $\big(U_0, T^{h_1}(U_0), \cdots, T^{h_n}(U_0)\big)$. Notice that it suffices to prove that, for any $(x_0, \cdots, x_n) \in I^{n+1}$, $H_{\boldsymbol{h}}$ is equal to (4.5). Indeed, we have

$$\begin{aligned} H_{\boldsymbol{h}}(x_0,\cdots,x_n) &= \mathbb{P}\big(U_0 \leq x_0, T^{h_1}(U_0) \leq x_1, \cdots, T^{h_n}(U_0) \leq x_n\big) \\ &= \mathbb{P}\Big(U_0 \in [0,x_0], U_0 \in \bigcup\nolimits_{k=1}^{s^{h_1}} \big(A^{\uparrow}_{h_1,k}(x_1) \bigcup A^{\downarrow}_{h_1,k}(x_1)\big), \cdots, U_0 \in \bigcup\nolimits_{k=1}^{s^{h_n}} \big(A^{\uparrow}_{h_n,k}(x_n) \bigcup A^{\downarrow}_{h_n,k}(x_n)\big)\Big) \\ &= \mathbb{P}\Big(U_0 \in [0,x_0] \bigcap\nolimits_{i=1}^{n} \bigcup\nolimits_{k=1}^{s^{h_i}} \big(A^{\uparrow}_{h_i,k}(x_i) \bigcup A^{\downarrow}_{h_i,k}(x_i)\big)\Big), \end{aligned}$$

which is precisely (4.5). The other assertion follows from Sklar's theorem in view of (4.9) and from the fact that $T^t(U_0) \stackrel{d}{=} U_0$, for all $t \in \mathbb{N}$. ■

Upon specializing the results seen so far, we can now determine the copulas associated to multidimensional vectors coming from a $\mathcal{T}_\varphi^\updownarrow$ process with φ increasing almost everywhere starting with the case $T\in\mathcal{T}^\updownarrow\backslash(\mathcal{T}^\uparrow\bigcup\mathcal{T}^\downarrow)$, which essentially embodies the other two cases.

Theorem 4.1. *Let* $T\in\mathcal{T}^\updownarrow\backslash(\mathcal{T}^\uparrow\bigcup\mathcal{T}^\downarrow)$, μ_T *be a* T*-invariant probability measure and* $\varphi\in\mathcal{L}^1(\mu_T)$ *be an almost everywhere increasing function. Let* $\{X_t\}_{t\in\mathbb{N}}$ *be the associated* $\mathcal{T}_\varphi^\updownarrow$ *process. If we let* $\{a_{h_n,k}\}_{k=0}^{s_{h_n}}$ *be the net associated to the nodes of* T^{h_n}, *and* $C_{\boldsymbol{t}}$ *be the copula associated to* $(X_t,X_{t+h_1},\cdots,X_{t+h_n})$, *for* $t,h_1,\cdots,h_n\in\mathbb{N}$, $0<h_1<\cdots<h_n$, *then*

$$\begin{aligned}C_{\boldsymbol{t}}(u_0,\cdots,u_n) &= \sum_{k\in n_0^\uparrow}\left[F_0\big(r^\uparrow_{h_n,k}(u_{1:n})\big)-F_0(a_{h_n,k-1})\right]+\sum_{k\in n_0^\downarrow}F_0(a_{h_n,k})-F_0\big(r^\downarrow_{h_n,k}(u_{1:n})\big)+\\ &\quad+\Big[\min\big\{u_0,F_0\big(r^\uparrow_{h_n,n_0}(u_{1:n})\big)\big\}-F_0(a_{h_n,n_0-1})\Big]\delta_{K^\uparrow_{h_n}}(n_0)+\\ &\quad+\max\big\{0,u_0-F_0\big(r^\downarrow_{h_n,n_0}(u_{1:n})\big)\big\}\,\delta_{K^\downarrow_{h_n}}(n_0),\end{aligned}$$

where $n_0=\big\{k:u_0\in\big(F_0(a_{h_n,k-1}),F_0(a_{h_n,k})\big]\big\}$, *with* $K^\uparrow_{h_n}$ *and* $K^\downarrow_{h_n}$ *given in* (2.1),

$$n_0^\uparrow=\{1,\cdots,n_0-1\}\bigcap K^\uparrow_{h_n},\quad n_0^\downarrow=\{1,\cdots,n_0-1\}\bigcap K^\downarrow_{h_n},$$

with $b^\uparrow_{h_n,k}$ *and* $b^\downarrow_{h_n,k}$ *given in* (4.4),

$$r^\uparrow_{h_n,k}(u_{1:n})=b^\uparrow_{h_n,k}\big(F_0^{-1}(u_{1:n})\big)\quad\text{and}\quad r^\downarrow_{h_n,k}(u_{1:n})=b^\downarrow_{h_n,k}\big(F_0^{-1}(u_{1:n})\big).\tag{4.10}$$

Proof: In view of Proposition 4.3, we only need to show the result for the vector $\big(U_0,T^{h_1}(U_0),\cdots,T^{h_n}(U_0)\big)$, where $h_1,\cdots,h_n\in\mathbb{N}$, $0<h_1<\cdots<h_n$. Considering $K^\uparrow_{h_n}$ and $K^\downarrow_{h_n}$ as in (2.1), let $b^\uparrow_{h_n,k}(x_{1:n})$, $b^\downarrow_{h_n,k}(x_{1:n})$, $\widetilde{A}^\uparrow_{h_n,k}(x_{1:n})$ and $\widetilde{A}^\downarrow_{h_n,k}(x_{1:n})$ be respectively as in (4.4), (4.2) and (4.3). Set $n_1=\big\{k:x_0\in(a_{h_n,k-1},a_{h_n,k}]\big\}$ and let $n_1^\uparrow=\{1,\cdots,n_1-1\}\bigcap K^\uparrow_{h_n}$ and $n_1^\downarrow=\{1,\cdots,n_1-1\}\bigcap K^\downarrow_{h_n}$. Notice that $n_1^\uparrow\bigcap n_1^\downarrow=\emptyset$ and $n_1^\uparrow\bigcup n_1^\downarrow=\{1,\cdots,n_1-1\}$. We can now rewrite (4.1), in view of Proposition 4.3 and setting $\boldsymbol{h}=(0,h_1,\cdots,h_n)$, as

$$\begin{aligned}H_{\boldsymbol{h}}(x_0,\cdots,x_n) &= \sum_{k\in n_1^\uparrow}\mu_T\Big(\big[a_{h_n,k-1},b^\uparrow_{h_n,k}(x_{1:n})\big]\Big)+\sum_{k\in n_1^\downarrow}\mu_T\Big(\big[b^\downarrow_{h_n,k}(x_{1:n}),a_{h_n,k}\big]\Big)+\\ &\quad+\mu_T\Big(\big[a_{n_1-1},\min\big\{x_0,b^\uparrow_{h_n,n_1}(x_{1:n})\big\}\big]\Big)\delta_{K^\uparrow_{h_n}}(n_1)+\\ &\quad+\mu_T\Big(\big[b^\downarrow_{h_n,n_1}(x_{1:n}),x_0\big]\Big)\,\delta_{[b^\downarrow_{h_n,n_1}(x_{1:n}),1]}(x_0)\delta_{K^\downarrow_{h_n}}e(n_1)\\ &= \sum_{k\in n_1^\uparrow}\left[F_0\big(b^\uparrow_{h_n,k}(x_{1:n})\big)-F_0(a_{h_n,k-1})\right]+\sum_{k\in n_1^\downarrow}\left[F_0(a_{h_n,k})-F_0\big(b^\downarrow_{h_n,k}(x_{1:n})\big)\right]+\\ &\quad+\Big[F_0\big(\min\big\{x_0,b^\uparrow_{h_n,n_1}(x_{1:n})\big\}\big)-F_0(a_{h_n,n_1-1})\Big]\delta_{K^\uparrow_{h_n}}(n_1)+\\ &\quad+\max\big\{0,F_0(x_0)-F_0\big(b^\downarrow_{h_n,n_1}(x_{1:n})\big)\big\}\,\delta_{K^\downarrow_{h_n}}(n_1)\end{aligned}$$

Now, by Sklar's Theorem and (3.1), taking $n_0=\big\{k:u_0\in\big(F_0(a_{h_n,k-1}),F_0(a_{h_n,k})\big]\big\}$, it follows that

$$\begin{aligned}C_{\boldsymbol{h}}(u_0,\cdots,u_n) &= H_{\boldsymbol{h}}\big(F_0^{-1}(u_0),\cdots,F_0^{-1}(u_n)\big)\\ &= \sum_{k\in n_0^\uparrow}\left[F_0\big(r^\uparrow_{h_n,k}(u_{1:n})\big)-F_0(a_{h_n,k-1})\right]+\sum_{k\in n_0^\downarrow}\left[F_0(a_{h_n,k})-F_0\big(r^\downarrow_{h_n,k}(u_{1:n})\big)\right]+\\ &\quad+\Big[\min\big\{u_0,F_0\big(r^\uparrow_{h_n,n_0}(u_{1:n})\big)\big\}-F_0(a_{h_n,n_0-1})\Big]\delta_{K^\uparrow_{h_n}}(n_0)+\\ &\quad+\max\big\{0,u_0-F_0\big(r^\downarrow_{h_n,n_0}(u_{1:n})\big)\big\}\,\delta_{K^\downarrow_{h_n}}(n_0),\end{aligned}$$

where $r^\uparrow_{h_n,k}$ and $r^\downarrow_{h_n,k}$ are as in the enunciate. ∎

From Proposition 4.1 we have all we need to work with the case $T \in \mathcal{T}^{\uparrow}\bigcup\mathcal{T}^{\downarrow}$. This is the content of the next proposition.

Proposition 4.4. *Let $T \in \mathcal{T}^{\uparrow}\bigcup\mathcal{T}^{\downarrow}$, μ_T be a T-invariant probability measure and $\varphi \in \mathcal{L}^1(\mu_T)$ be an almost everywhere increasing function. Let $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}_\varphi^{\uparrow}$ or $\mathcal{T}_\varphi^{\downarrow}$ process. If we let $\{a_{h_n,k}\}_{k=0}^{s_{h_n}}$ be the net associated to the nodes of T^{h_n}, and $C_{\boldsymbol{t}}$ be the copula associated to $(X_t, X_{t+h_1}, \cdots, X_{t+h_n})$, for $t, h_1, \cdots, h_n \in \mathbb{N}$, $0 < h_1 < \cdots < h_n$, then*

(i). *if $T \in \mathcal{T}^{\uparrow}$,*

$$C_{\boldsymbol{t}}(u_0, \cdots, u_n) = \sum_{k=1}^{n_0-1} \big[F_0\big(r_{h_n,k}(u_{1:n})\big) - F_0(a_{h_n,k-1})\big] + \min\big\{u_0, F_0\big(r_{h_n,n_0}(u_{1:n})\big)\big\} - F_0(a_{h_n,n_0-1});$$

(ii). *if $T \in \mathcal{T}^{\downarrow}$,*

$$C_{\boldsymbol{t}}(u_0, \cdots, u_n) = \sum_{k=1}^{n_0-1} \big[F_0(a_{h_n,k}) - F_0\big(r_{h_n,k}(u_{1:n})\big)\big] + \max\big\{0, u_0 - F_0\big(r_{h_n,n_0}(u_{1:n})\big)\big\},$$

where $n_0 := \big\{k : u_0 \in \big(F_0(a_{h_n,k-1}), F_0(a_{h_n,k})\big]\big\}$, $b_{h_n,k}^{\uparrow}$ and $b_{h_n,k}^{\downarrow}$ given in (4.4), and

$$r_{h_n,k}(u_{1:n}) := \begin{cases} b_{h_n,k}^{\uparrow}\big(F_0^{-1}(u_{1:n})\big), & \text{if } \ T \in \mathcal{T}^{\uparrow}, \\ b_{h_n,k}^{\downarrow}\big(F_0^{-1}(u_{1:n})\big), & \text{if } \ T \in \mathcal{T}^{\downarrow}. \end{cases}$$

Proof: We observe that $r_{h_n,k}$ is just a combination of the two expressions in (4.10) and that if $T \in \mathcal{T}^{\uparrow}$, then $K_{h_n}^{\downarrow} = \emptyset$, so that $n_0^{\uparrow} = \{1, \cdots, n_0 - 1\}$ and $n_0^{\downarrow} = \emptyset$, while if $T \in \mathcal{T}^{\downarrow}$, then $K_{h_n}^{\uparrow} = \emptyset$ so that $n_0^{\downarrow} = \{1, \cdots, n_0 - 1\}$ and $n_0^{\uparrow} = \emptyset$. With this in mind, the results follow directly from Theorem 4.1. ■

The generalization to the n-dimensional case for φ an almost surely decreasing function leads to more complicated formulas in terms of the copulas in Theorem 4.1 and Proposition 4.4. Although the set up here is much more general than the one in Lopes and Pumi (2013), it is interesting to notice that the same result valid there can be applied here, so that we include it just for the sake of completeness.

Proposition 4.5. *Let $T \in \mathcal{T}^{\updownarrow}$, μ_T be a T-invariant probability measure and let $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}_\varphi^{\updownarrow}$ process for $\varphi \in \mathcal{L}^1(\mu_T)$ an almost surely decreasing function. Let $t, h_1, \cdots, h_n \in \mathbb{N}$, $0 < h_1 < \cdots < h_n$ and set $Y_k := T^{h_k}(U_0)$ and $Y_0 := U_0$. If we denote the copula associated to $(X_t, X_{t+h_1}, \cdots, X_{t+h_n})$ by $C_{\boldsymbol{t}}$, then the following relation holds*

$$\begin{aligned} C_{\boldsymbol{t}}(u_0, \cdots, u_n) = 1 - n + \sum_{i=0}^{n} u_i + \sum_{i=0}^{n}\sum_{j=i+1}^{n} C_{Y_i,Y_j}(1-u_i, 1-u_j) + \cdots + \\ + (-1)^{n-1} \sum_{k_1=0}^{n}\sum_{k_2=k_1+1}^{n} \cdots \sum_{k_{n-1}=k_{n-2}+1}^{n} C_{Y_{k_1},\cdots,Y_{k_{n-1}}}(1-u_{k_1}, \cdots, 1-u_{k_{n-1}}) + \\ + (-1)^n C_{U_0,Y_1,\cdots,Y_n}(1-u_0, \cdots, 1-u_n), \end{aligned} \tag{4.11}$$

everywhere in I^{n+1}.

Proof: See Proposition 4.1 in Lopes and Pumi (2013). ■

The copula in Proposition 4.5 can be explicitly calculated since (4.11) is written as sums of the copulas of vectors containing U_0 and $T^t(U_0)$ for different t's. Hence, Theorem 2.1 can be applied to derive the desired formulas in terms of the copulas in Theorem 4.1 and Proposition 4.4.

Remark 4.1. In principle, we could have assumed T belonging to the slightly broader family of $C^1_{1\text{-}1}$ transformations. However, for $T \in C^1_{1\text{-}1} \setminus C^{1+\alpha}_{1\text{-}1}$, if an absolutely continuous T-invariant probability measure exists, it does not have a positive continuous Radon-Nikodym derivative. This implies that F_0^{-1} is not always well defined. In this case one can substitute F_0^{-1} by its pseudo-inverse, $F_0^{(-1)}$. Since $F_0 \circ F_0^{(-1)}$ and $F_0^{(-1)} \circ F_0$ are not the identity map, the expressions for the copulas become more complex. If $T \in C^1_{1\text{-}1} \setminus C^{1+\alpha}_{1\text{-}1}$, the results in Sections 3 and 4 hold if F_0^{-1} is substituted by $F_0^{(-1)}$ and expressions of the form $F_0\big(F_0^{(-1)}(x)\big)$ and $F_0^{(-1)}\big(F_0(x)\big)$ will appear instead of the identities $F_0\big(F_0^{-1}(x)\big) = x$ and $F_0^{-1}\big(F_0(x)\big) = x$.

5 Numerical Approximations and Computational Issues

Let $T \in \mathcal{T}^{\updownarrow}$, $h > 0$, μ_T be a T-invariant probability measure and consider the associated $\mathcal{T}^{\updownarrow}_{\varphi}$ process for $\varphi \in \mathcal{L}^1(\mu_T)$ a monotone function. The computation and implementation of the copulas derived in Sections 3 and 4 depend on the knowledge of several elements. These elements are the invariant probability measure μ_T, the associated distribution function F_0 and its inverse F_0^{-1}, the inverse of T^h in each branch $(\{\mathcal{T}_{h,k}\}_{k=1}^{s^h})$ and the net $\{a_{h,k}\}_{k=0}^{s^h}$ associated to T^h, where $s > 1$ denotes, as usual, the number of branches of T.

As mentioned in Section 2, the general problem of determining the existence of a T-invariant absolutely continuous probability measure is usually a hard one. Therefore, one rarely finds an explicit formula for a T-invariant probability measure. Furthermore, although the computation of $T^h(y)$ is usually straightforward, the exact calculation of $\mathcal{T}_{h,k}(y)$ can be a highly complex task. Even for small values of h, the apparently simple calculation of the net associated to T^h can be troublesome. Nevertheless, one can still rely on approximations in order to implement, compute and study the copulas related to $\mathcal{T}^{\updownarrow}_{\varphi}$ processes.

With this in mind, our goal in this section is to present general results and conditions to construct simple approximations to the copulas derived in Sections 3 and 4 in such a way to guarantee its uniform convergence to the true copula.

5.1 Approximating the T-invariant probability measure and related functions

Perhaps the most appealing way to approximate a measure, assuming it is ergodic, is by using Birkhoff's ergodic theorem. Recall that a measure μ is called a *Sinai-Bowen-Ruelle measure* (SBR, for short) if the weak convergence

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{T^k(x)}(A) \longrightarrow \mu(A) \tag{5.1}$$

holds for almost all $x \in I$ and all μ-continuity sets A. In this work, all the T-invariant probability measure discussed are assumed to be absolutely continuous with respect to the Lebesgue measure. However, if μ_T is ergodic and absolutely continuous with respect to the Lebesgue measure, it is also an SBR measure (see, for instance, Keller, 1998), so that we will only consider the latter more general case.

Given $T \in \mathcal{T}^{\updownarrow}$ and μ_T a T-invariant absolutely continuous SBR probability measure, one way to approximate μ_T is by truncating expression (5.1) for a reasonably large value of $n > 1$. That is, let $x_0 \in I$ be a point such that the weak convergence (5.1) holds. Set

$$\mu_n(A; T, x_0) := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{T^k(x_0)}(A), \tag{5.2}$$

for all μ_T-continuity sets A. This is not the only way to approximate an SBR measure, see for instance Lopes and Pumi (2013) and references therein. For simplicity and since no confusion will arise, we shall drop T and x_0 from the notation of μ_n.

From μ_n, an approximation for F_0 is obtained simply by setting $\hat{F}_n(x) = \mu_n\big([0,x];T,x_0\big)$. From a computational standpoint, fixed x_0 and truncation point $n > 0$, to calculate μ_n for different sets one needs to compute the iteration vector $\big(x_0, T(x_0), \cdots, T^{n-1}(x_0)\big)$ only once. Furthermore, by the choice of x_0, $T^i(x_0) \neq T^j(x_0)$ for all $i \neq j$, therefore, $\hat{F}_n$ will coincide with the empirical distribution function based on the iteration vector. To improve the performance pointwisely, one can use some interpolation method based on the iteration vector and $\hat{F}_n$. That is, we choose a sequence of interpolating functions[4] based on $\big(x_0, T(x_0), \cdots, T^{n-1}(x_0)\big)$ and its image by $\hat{F}_n$.

For approximations based on a vector of iterations, the limits taken are understood to be in terms of partitions in the following manner. Starting with a set of points $R_m = \{x_1, \cdots, x_m\}$, we consider refinements obtained by adding a single point to the set R_m. That is, we consider refinements of the form $R_{m+1} = R_m \bigcup \{x_{m+1}\}, \cdots, R_{m+k} = R_{m+k-1} \bigcup \{x_{m+k}\}$. Let $f_m(\cdot) := f(\,\cdot\,; R_m)$ be an approximation based on R_m. For a sequence of refinements $\{R_k\}_{k=m+1}^{\infty}$ we consider the sequence of functions $\{f_k\}_{k=m+1}^{\infty}$. If the sequence $\{f_k\}_{k=0}^{\infty}$ has a limit, we set $\lim\limits_{m\to\infty} f_m(\cdot) := \lim\limits_{m\to\infty} f(\,\cdot\,; R_m)$.

Proposition 5.1. *Let $S = \{x_1, \cdots, x_n\}$ be a given (ordered) sample of some continuous and monotone distribution F_0 and let $\hat{F}_n$ be the empirical distribution based on S. Let F_n be an approximation based on a sequence of interpolating functions defined from S and let $x_0 = 0$ and $x_{n+1} = 1$. Suppose that F_n satisfies, for each $x \in (x_i, x_i + 1)$, $F_n(x) \in \big(\hat{F}_n(x_i), \hat{F}_n(x_{i+1})\big)$, $i = 0, \cdots, n$ and $F_n(x_i) = \hat{F}_n(x_i)$, for all $i = 0, \cdots, n+1$. Then $F_n(x) \to F_0(x)$ uniformly over $x \in I$. If, in addition, F_n is continuous and monotone, then also $F_n^{-1}(x) \to F_0^{-1}(x)$ uniformly over $x \in I$.*

Proof: By the Glivenko-Cantelli theorem, $\widehat{F}_n(x) \to F_0(x)$ uniformly in $x \in [0,1]$, so that, given $\varepsilon > 0$, there exists $n_0 > 0$ depending on ε only such that $\big|\widehat{F}_n(x) - F_0(x)\big| < \varepsilon/2$, for all $x \in I$, whenever $n > n_0$. Now, for $x \in [0,1)$, there exists $k \in \{1, \cdots, n\}$ such that $x \in [x_k, x_{k+1})$ and since $F_n(x) \in \big[\hat{F}_n(x_i), \hat{F}_n(x_{i+1})\big)$, if $n > \max\{n_0, \lceil 2/\varepsilon \rceil\}$, it follows that

$$\begin{aligned} \big|F_n(x) - F_0(x)\big| &\leq \big|F_n(x) - \widehat{F}_n(x)\big| + \big|\widehat{F}_n(x) - F_0(x)\big| < \big|\widehat{F}_n(x_{k+1}) - \widehat{F}_n(x_k)\big| + \frac{\varepsilon}{2} \\ &\leq \sup_{i=1,\cdots,n-1} \big\{\big|\widehat{F}_n(x_{i+1}) - \widehat{F}_n(x_i)\big|\big\} + \frac{\varepsilon}{2} \;\leq\; \frac{1}{n} + \frac{\varepsilon}{2} \;<\; \varepsilon, \end{aligned}$$

uniformly in $x \in [0,1)$ and, by the continuity of F_0, it holds uniformly in I.

As for the inverse, notice that, by hypothesis, F_n^{-1} is (uniformly) continuous and monotone. Therefore, given $\varepsilon > 0$ and $y \in I$, there exists a $\delta > 0$ (depending on ε only) such that

$$|x - y| < \delta \implies |F_n^{-1}(x) - F_n^{-1}(y)| < \varepsilon.$$

By the Glivenko-Cantelli theorem, there exists $n_1 > 0$ (depending on δ only) such that $\big|F_n(x) - F_0(x)\big| < \delta$, whenever $n > n_1$, and the inequality holds for all $x \in [0,1]$. Furthermore, the monotonicity of F_0 implies the existence of $z_0 \in [0,1]$ such that $y = F_0(z_0)$. Now, taking $n > n_1$, it follows that

$$\big|F_n^{-1}(y) - F_0^{-1}(y)\big| = \big|F_n^{-1}\big(F_0(z_0)\big) - z_0\big| = \big|F_n^{-1}\big(F_0(z_0)\big) - F_n^{-1}\big(F_n(z_0)\big)\big| < \varepsilon,$$

and since n_1 does not depend on y, the convergence is uniform. ■

[4]by interpolation function, we mean a function $f : [a,b] \to [c,d]$ such that, given a collection of pairs $P = \{(u_1, v_1), \cdots, (u_n, v_n)\}$, $u_i \in [a,b]$ and $v_i \in [c,d]$, $i = 1, \cdots, n$, and a point $x \in [a,b]$, f assigns a value $f(x;P)$ for each $x \in [a,b] \setminus \{u_1, \cdots, u_n\}$ and $f(u_i;P) = v_i$, for $i = 1, \cdots, n$. A simple linear interpolation or a spline interpolation are examples of such functions.

Suppose that T has $s > 1$ nodes. There is no general optimal way of approximating the net related to T^h, for a given $h > 1$, so the task has to be handled case by case. A general method, which may not be optimal in most cases, is as follows. Let $S := \{x_1, \cdots, x_m\}$ be an ordered set of points in I and, to avoid trivialities, suppose that $x_1 = 0$, $x_m = 1$ and $x_i \neq x_j$, for $i, j = 1, \cdots, m$, $i \neq j$. Let R denote the image of S by T^h. The next step is to identify possible intervals where the discontinuities of T^h lie on, by using the elements of R. In order to do that, S must reflect all the s^h discontinuities of T^h. Usually a relatively thin grid of equally spaced points solves the problem (obviously, $m > s^h$ points should be used).

If $T \in \mathcal{T}^{\uparrow}$, the identification can be done in the following manner. Note first that $a_{h,0} = 0$ and $a_{h,s^h} = 1$, for any h. Let $D^+ = \big\{i : T^h(x_i) > T^h(x_{i+1})\big\} \subset \{1, \cdots, m\}$ and let $\{d_j\}_{j=1}^{s^h-1}$ denote the ordered elements of D^+. The set D^+ contains the indexes $i \in \{1, \cdots, m\}$ for which the interval $[x_{d_k}, x_{d_{k+1}}]$ contains the k-th discontinuity of T^h. If $T \in \mathcal{T}^{\downarrow}$, we let $D^- = \big\{i : T^h(x_i) < T^h(x_{i+1})\big\}$. If $T \in \mathcal{T}^{\updownarrow} \setminus (\mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow})$, one has to define D as a mixture of D^+ and D^-. One obvious way to detect a poor choice of S is the number of elements in the respective set D^+, D^- or D, which should have exactly $s^h - 1$ distinct points.

Once the intervals containing the discontinuities of T^h are found, one can perform a simple bisection, a golden search or any other root-finder algorithm to find an approximation to $a_{h,k}$ up to a predetermined acceptable error. This gives an "instantaneous" estimate which, in terms of precision, should not improve as the number of elements in S increases. However, given the fact that, depending on the transformation, most of the times there are easier ways of calculating an approximation to $a_{h,k}$ based on S, our results cover this possibility as well. See also Remark 5.1.

As an example of this possibility, we mention the method presented in Lopes and Pumi (2013). Although presented in the context of Manneville-Pomeau transformations, it can be extended to a broader family of transformations as follows. Consider transformations of the type $T(x) = g(x) \pmod 1$ for a suitable increasing and differentiable function g, such that T has $s > 1$ full branches. Let S and D^+ be as before, and consider the function $T^*_{i,h} : [x_{d_i}, x_{d_i+1}] \to [0, 2]$ given by $T^*_{i,h}(x) := g\big(T^{h-1}(x)\big)$, for $i = 1, \cdots, s^h - 2$. Clearly $T^*_{i,h}(x_{d_i}) < 1$, $T^*_{i,h}(x_{d_i+1}) > 1$ and

$$T^h_s(x) = T^*_{i,h}(x) - \delta_{[1,2]}\big(T^*_{i,h}(x)\big),$$

for all $x \in [x_{d_i}, x_{d_i+1}]$. We denote the approximation to $a_{h,i}$, based on S, by $a^m_{h,i}$, and define it as the linear interpolation of 1 between the points $\big(x_{d_i}, T^*_{i,h}(x_{d_i})\big)$ and $\big(x_{d_i+1}, T^*_{i,h}(x_{d_i+1})\big)$. This approximation can be very crude if m is small, but the precision increases fast as m increases, especially for large h. The method can be easily adapted to cover the cases where g is decreasing and differentiable. In Section 6 we present a simpler example.

As for the approximation to $\mathcal{T}_{h,k}$, we shall use an interpolation argument based on a grid of values from T^h. Let $S = \{x_1, x_2, \cdots, x_m\}$, be an ordered set of points in I and assume that $x_1 = 0$, $x_m = 1$ and $x_i \neq x_j$, for $i, j = 1, \cdots, m$, $i \neq j$. Suppose the net $\{a_{h,k}\}_{k=0}^{s^h}$ related to T^h is known or estimates based on S, say $\{a^m_{h,k}\}_{k=0}^{s^h}$, are available and satisfy, for all $k = 0, \cdots, s^h$, $a^m_{h,k} \to a_{h,k}$, as m tends to infinity. Let D^+, D^- or D, depending on the case, be given as before. For $i = 1, \cdots, s^h$, let

$$S_{h,i} := \{x^{(1)}_{h,i}, \cdots, x^{(p_i)}_{h,i}\} := \{a^m_{h,i-1}, x_{d_i+1}, \cdots, x_{d_{i+1}}, a^m_{h,i}\}, \tag{5.3}$$

$$R_{h,i} := \{y^{(1)}_{h,i}, \cdots, y^{(p_i)}_{h,i}\} := \big\{0, T^h(x_{d_i+1}), \cdots, T^h(x_{d_{i+1}}), 1\big\}, \tag{5.4}$$

and $O_{h,i} := \big\{(y^{(k)}_{h,i}, x^{(k)}_{h,i})\big\}_{k=1}^{p_i}$, where $p_i := d_{i+1} - d_i + 2$. Notice that the elements on $S_{h,i}$ are just the values among $x_1, \cdots, x_m$ that lie on the i-th node of T^h and that $R_{h,i}$ is the image of $S_{h,i}$ by T^h. Suppose that a sequence of interpolation functions $h_{i,m} : I \to [a^m_{h,i-1}, a^m_{h,i}]$, $i = 1, \cdots, s^h$, is given. Given $y \in [0, 1] \setminus \{T^h(x_1), \cdots, T^h(x_m)\}$, there exists $y^{(k)}_{h,i} \in R_{h,i}$ such that $y \in (y_{h,i}^{(k)}, y_{h,i}^{(k+1)})$. We define an approximation to $\mathcal{T}_{h,k}(y)$ based on S, which we denote by $\mathcal{T}_{h,i}^{m}(y)$, simply by

$$\mathcal{T}_{h,i}^{m}(y) := h_{i,m}(y; O_{h,i}), \tag{5.5}$$

for $i = 1, \cdots, s^h$.

Remark 5.1. The estimates $a_{h,k}^m$ and $\mathcal{T}_{h,k}^m$ are both assumed to be based on S. This is convenient because it allows the following updating scheme to determine both approximations: given a refinement of S, first we update the net, and secondly, the net new values are used to determine the approximation to $\mathcal{T}_{h,k}$. This type of scheme can be applied very often in practice. The proofs of the results in this section, however, assume solely that $a_{h,k}^m \to a_{h,k}$, as m goes to infinity. The inverse scheme (update $\mathcal{T}_{h,k}^m$ and then update $a_{h,k}^m$) also works, because $a_{h,k} = \mathcal{T}_{h,k}(1)$, for $k = 1, \cdots, s^h$, but it is computationally more delicate to implement than the former. This happens because in updating $\mathcal{T}_{h,k}^m$ to, say, $\mathcal{T}_{h,k}^{m+1}$, one uses $a_{h,k}^m$. Next, to obtain $a_{h,k}^{m+1}$, $T_{h,k}^{m+1}$ is applied which, by its turn, is based on the old value $a_{h,k}^m$. To account for this discrepancy, a posterior adjustment has to be made, but we shall not get into details here.

Proposition 5.2. *Let $T \in \mathcal{T}^{\updownarrow}$ and suppose T has $s > 1$ branches. Let $S = \{x_1, x_2, \cdots, x_m\}$, be an ordered set of points in I and assume that $x_1 = 0$, $x_m = 1$ and $x_i \neq x_j$, for $i, j = 1, \cdots, m$, $i \neq j$. For $h > 0$, let $\{a_{h,j}^m\}_{j=0}^{s^h}$ be approximations to $\{a_{h,j}\}_{j=0}^{s^h}$ based on S such that $a_{h,j}^m \to a_{h,j}$, for all $j = 0, \cdots, s^h$, as $m \to \infty$. Let $S_{h,i}$ and $R_{h,i}$ be as in* (5.3) *and* (5.4), *respectively. For a sequence of interpolation functions $\{h_{i,m}\}_{i=1}^{s^h}$ based on S, let $\mathcal{T}_{h,i}^m$ be as in* (5.5)*. If, for all $y \in (y_{h,i}^{(k)}, y_{h,i}^{(k+1)})$, $\mathcal{T}_{h,i}^m(y) \in (x_{h,i}^{(k)}, x_{h,i}^{(k+1)})$, for $i = 1, \cdots, s^h$ and $k \in \{1, \cdots, p_i - 1\}$, then $\mathcal{T}_{h,i}^m \longrightarrow \mathcal{T}_{h,i}$ uniformly, as $m \to \infty$.*

Proof: The uniform continuity of $\mathcal{T}_{h,i}$ implies that, for any given $\varepsilon > 0$, there exists a $\delta > 0$ depending only on ε, such that

$$|x - y| < \delta \implies \big|\mathcal{T}_{h,i}(x) - \mathcal{T}_{h,i}(y)\big| < \varepsilon, \tag{5.6}$$

for all $x \in I$. Let $S := \{x_1, \cdots, x_{m_0}\}$ be an ordered set of points in I and assume that $x_1 = 0$, $x_{m_0} = 1$ and $x_i \neq x_j$, $i, j = 1, \cdots, m_0$, $i \neq j$. Without loss of generality, assume that $m_0 > 0$ is such that

$$\sup_{i=1,\cdots,m_0-1} \big\{|x_{i+1} - x_i|\big\} < \delta. \tag{5.7}$$

For $m > m_0$, let $S_m = \{x_1^*, \cdots, x_m^*\} \supset S$ be a size m refinement of S. By hypothesis, given $y \in (0,1)$, for each $i \in \{1, \cdots, s^h\}$, there exists a $k \in \{1, \cdots, p_i - 1\}$ such that $\mathcal{T}_{h,i}^m(y) \in (y_{h,i}^{(k)}, y_{h,i}^{(k+1)}) = \big(\mathcal{T}_{h,i}(x_{h,i}^{(k)}), \mathcal{T}_{h,i}(x_{h,i}^{(k+1)})\big)$. Also, since $T \in \mathcal{T}^{\updownarrow}$, $\mathcal{T}_{h,i}\big(x_{h,i}^{(k)}\big) \leq \mathcal{T}_{h,i}(y) < \mathcal{T}_{h,i}\big(x_{h,i}^{(k+1)}\big)$, so that, by (5.6) and (5.7),

$$\begin{aligned} \big|\mathcal{T}_{h,i}^m(y) - \mathcal{T}_{h,i}(y)\big| &\leq \big|\mathcal{T}_{h,i}\big(x_{h,i}^{(k+1)}\big) - \mathcal{T}_{h,i}\big(x_{h,i}^{(k)}\big)\big| \\ &\leq \sup_{j=1,\cdots,m-1} \big\{\big|\mathcal{T}_{h,i}(x_{j+1}) - \mathcal{T}_{h,i}(x_j)\big|\big\} < \varepsilon, \end{aligned}$$

independently of $y \in (0,1)$. For $y \in \{0,1\}$, by definition $\mathcal{T}_{h,i}^m(y) = \mathcal{T}_{h,i}(y)$, so that the result holds uniformly for all $y \in I$. ■

5.2 Approximating the copulas

In this subsection we present a general approximation for the copulas in Section 3 and prove its uniform convergence to the true copula. As mentioned before, the formulas for the copulas presented in Remark 3.5 are the ones used for approximation purposes. Although we could proceed by using the approximations developed in the last subsection, our convergence results hold in a more general setting. Let $T \in \mathcal{T}^{\updownarrow} \setminus (\mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow})$ and μ_T be a T-invariant probability measure. Let $\varphi \in \mathcal{L}^1(\mu_T)$ and $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}_\varphi^{\updownarrow}$ process. Let μ_n be an approximation to μ_T and let F_n and F_n^{-1} be approximations to F_0 and F_0^{-1}, respectively. Suppose T has $s > 1$ nodes and, for $h > 0$, let $\{a_{h,i}^m\}_{i=0}^{s^h}$ be an approximation to the net related to T^h. Also let $\mathcal{T}_{h,i}^m$ be an approximation to $\mathcal{T}_{h,i}$, $i = 1, \cdots, s^h$. Define, for all $u, v \in I$,

$$\begin{aligned} C_{m,n}^{\updownarrow}(u,v;h) \;=\; & \sum_{k\in n_0^{\uparrow}} \mu_n\Big(\big[a_{h,k-1}^m, \mathcal{T}_{h,k}^m\big(F_n^{-1}(v)\big)\big]\Big) + \sum_{k\in n_0^{\downarrow}} \mu_n\Big(\big[\mathcal{T}_{h,k}^m\big(F_n^{-1}(v)\big), a_{h,k}^m\big]\Big) + \\ & + \mu_n\Big(\big[a_{h,\widehat{n}_0-1}^m, \min\big\{F_n^{-1}(u), \mathcal{T}_{h,\widehat{n}_0}^m\big(F_n^{-1}(v)\big)\big\}\big]\Big)\delta_{K_h^{\uparrow}}(\widehat{n}_0) + \\ & + \mu_n\Big(\big[\mathcal{T}_{h,\widehat{n}_0}^m\big(F_n^{-1}(v)\big), F_n^{-1}(u)\big]^+\Big)\delta_{K_h^{\downarrow}}(\widehat{n}_0), \end{aligned} \tag{5.8}$$

where $\widehat{n}_0 = n_0(m,n) = \big\{k : u \in \big[F_n(a_{h,k}^m), F_n(a_{h,k+1}^m)\big)\big\}$. Expression (5.8) can be used as an approximation to $C_{X_t,X_{t+h}}^{\updownarrow}$ in (3.6), for $t \geq 0$. In the next theorem, we show that, under certain simple conditions on the approximations in (5.8), $C_{m,n}^{\updownarrow}(u,v) \to C_{X_t,X_{t+h}}^{\updownarrow}(u,v)$ uniformly in $(u,v) \in I^2$ as m and n go to infinity.

Theorem 5.1. *Let $T \in \mathcal{T}^{\updownarrow} \setminus (\mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow})$ and μ_T be a T-invariant probability measure. Let $\varphi \in \mathcal{L}^1(\mu_T)$ be an almost everywhere increasing function and let $\{X_t\}_{t\in\mathbb{N}}$ be the $\mathcal{T}_\varphi^{\updownarrow}$ associated process. Let μ_n be a sequence of measures converging weakly to μ_T and let F_n and F_n^{-1} be approximations to F_0 and F_0^{-1}, respectively, such that $F_n \to F_0$ and $F_n^{-1} \to F_0^{-1}$ uniformly. Suppose T has $s > 1$ nodes and, for $h > 1$, let $\{a_{h,i}^m\}_{i=0}^{s^h}$ be an approximation to the net related to T^h such that $a_{h,i}^m \to a_{h,i}$, for all $i = 0, \cdots, s^h$, as m goes to infinity. Also let $\{\mathcal{T}_{h,i}^m\}_{i=1}^{s^h}$ be an approximation to $\{\mathcal{T}_{h,i}\}_{i=1}^{s^h}$ and suppose $\mathcal{T}_{h,i}^m \to \mathcal{T}_{h,i}$ uniformly for all $i = 1, \cdots, s^h$. Let $C_{m,n}^{\updownarrow}$ be given by (5.8) with the approximations just defined. Then, for all $(u,v) \in I^2$,*

$$\lim_{n\to\infty}\lim_{m\to\infty} C_{m,n}^{\updownarrow}(u,v;h) = \lim_{m\to\infty}\lim_{n\to\infty} C_{m,n}^{\updownarrow}(u,v;h) = \lim_{m,n\to\infty} C_{m,n}^{\updownarrow}(u,v;h),$$

and the common limit is $C_{X_t,X_{t+h}}^{\updownarrow}(u,v)$ given by (3.6), for all $t > 0$. Furthermore, the limits above are uniform in (u,v).

Proof: The hypothesis on the approximations in expression (5.8), allow us to apply Lemma 5.1 in Lopes and Pumi (2013) to conclude that, for any k,

$$\lim_{m,n\to\infty} \mu_n\Big(\big[a_{h,k-1}^m, \mathcal{T}_{h,k}^m\big(F_n^{-1}(v)\big)\big]\Big) = \mu_T\Big(\big[a_{h,k-1}, \mathcal{T}_{h,k}\big(F_0^{-1}(v)\big)\big]\Big),$$

and, by the same argument,

$$\lim_{m,n\to\infty} \mu_n\Big(\big[\mathcal{T}_{h,k}^m\big(F_n^{-1}(v)\big), a_{h,k}^m\big]\Big) = \mu_T\Big(\big[\mathcal{T}_{h,k}\big(F_0^{-1}(v)\big), a_{h,k}\big]\Big).$$

From the same lemma we also conclude that the iterated limits exist and are equal to the double ones above and all limits are uniform in v. By using the same argument as in the proof of Theorem 5.1 in Lopes and Pumi (2013), we conclude that

$$\lim_{m,n\to\infty} \mu_n\Big(\big[a_{h,\widehat{n}_0-1}^m, \min\big\{F_n^{-1}(u), \mathcal{T}_{h,\widehat{n}_0}^m\big(F_n^{-1}(v)\big)\big\}\big]\Big) = \mu_T\Big(\big[a_{h,n_0-1}, \min\big\{F_0^{-1}(u), \mathcal{T}_{h,n_0}\big(F_0^{-1}(v)\big)\big\}\big]\Big),$$

and that the iterated limits are equal to the double one above. Furthermore, the limits are all uniform in $u, v \in I$. So, it remains to show that

$$\lim_{m,n\to\infty} \mu_n\Big(\big[\mathcal{T}_{h,\widehat{n}_0}^m\big(F_n^{-1}(v)\big), F_n^{-1}(u)\big]^+\Big) = \mu_T\Big(\big[\mathcal{T}_{h,n_0}\big(F_0^{-1}(v)\big), F_0^{-1}(u)\big]^+\Big),$$

uniformly and that the iterated limits are equal to the double one. First, the existence and the uniform convergence of the iterated limits follow by the same argument in the proof of Lemma 5.1 in Lopes and Pumi (2013). As for the double limit, by the hypothesis on μ_T, the Radon-Nikodym theorem applies and implies the existence of a non-negative bounded (since we are restricted to I) continuous function h such that, for any $A \in \mathcal{B}(I)$,

$$\mu_T(A) = \int_A h(x)\mathrm{d}\lambda \leq M\lambda(A),$$

where $M = \sup_{x\in I}\{h(x)\} < \infty$. Now, the uniform convergence of F_n^{-1} to F_0^{-1} implies the existence of $n_0 > 0$ such that, if $n > n_0$, $F_n^{-1}(u) \in K(\varepsilon) := [F_0^{-1}(u) - \varepsilon/10M, F_0^{-1}(u) + \varepsilon/10M]$ and

$$\mu_T\big(K(\varepsilon)\big) \leq M\lambda\big(K(\varepsilon)\big) = \frac{\varepsilon}{5},$$

for all $u \in I$. The rest of the proof is carried out by mimicking the proof of Lemma 5.1 in Lopes and Pumi (2013) upon substituting $K_1(\varepsilon)$ there by $K(\varepsilon)$ just defined and noticing that n_0 here plays the role of m_1 there. This completes the proof of the theorem. ∎

From Theorem 5.1, we derive approximations to the copulas in the case $T \in \mathcal{T}^{\updownarrow} \setminus (\mathcal{T}^{\uparrow}\bigcup\mathcal{T}^{\downarrow})$ given in (3.8) and (3.9).

Corollary 5.1. *Let $T \in \mathcal{T}^{\uparrow}\bigcup\mathcal{T}^{\downarrow}$ and μ_T be a T-invariant probability measure. Let $\varphi \in \mathcal{L}^1(\mu_T)$ be an almost everywhere increasing function and let $\{X_t\}_{t=0}^{\infty}$ be the $\mathcal{T}_{\varphi}^{\downarrow}$ or $\mathcal{T}_{\varphi}^{\uparrow}$ associated process. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of measures converging weakly to μ_T and let F_n and F_n^{-1} be approximations to F_0 and F_0^{-1}, respectively, such that $F_n \to F_0$ and $F_n^{-1} \to F_0^{-1}$ uniformly. Suppose T has $s > 1$ nodes and, for $h > 1$, let $\{a_{h,i}^m\}_{i=0}^{s^h}$ be an approximation to the net related to T^h such that $a_{h,i}^m \to a_{h,i}$, for all $i = 0, \cdots, s^h$, as m goes to infinity. Also let $\{\mathcal{T}_{h,i}^m\}_{i=1}^{s^h}$ be an approximation to $\{\mathcal{T}_{h,i}\}_{i=1}^{s^h}$ and suppose $\mathcal{T}_{h,i}^m \to \mathcal{T}_{h,i}$ uniformly for all $i = 1, \cdots, s^h$.*

(i). *If $T \in \mathcal{T}^{\uparrow}$, then, for all $(u, v) \in I^2$,*

$$\begin{aligned} C_{m,n}^{\uparrow}(u, v; h) \;=\; & \sum_{k=1}^{\widehat{n}_0-1} \mu_n\Big(\big[a_{h,k-1}^m, \mathcal{T}_{h,k}^m\big(F_n^{-1}(v)\big)\big]\Big)\delta_{\mathbb{N}^*}(\widehat{n}_0 - 1) + \\ & + \mu_n\Big(\big[a_{h,\widehat{n}_0-1}^m, \min\big\{F_n^{-1}(u), \mathcal{T}_{h,\widehat{n}_0}^m\big(F_n^{-1}(v)\big)\big\}\big]\Big) \end{aligned}$$

is an approximation to the copula (3.8),

$$\lim_{n\to\infty}\lim_{m\to\infty} C_{m,n}^{\uparrow}(u, v; h) = \lim_{m\to\infty}\lim_{n\to\infty} C_{m,n}^{\uparrow}(u, v; h) = \lim_{m,n\to\infty} C_{m,n}^{\uparrow}(u, v; h),$$

and the common limit is $C_{X_t,X_{t+h}}^{\uparrow}(u, v)$ given by (3.8).

(ii). *If $T \in \mathcal{T}^{\downarrow}$, then, for all $(u, v) \in I^2$,*

$$C_{m,n}^{\downarrow}(u, v; h) = \sum_{k=1}^{\widehat{n}_0-1} \mu_n\Big(\big[\mathcal{T}_{h,k}^m\big(F_n^{-1}(v)\big), a_{h,k}^m\big]\Big)\delta_{\mathbb{N}^*}(\widehat{n}_0 - 1) + \mu_n\Big(\big[\mathcal{T}_{h,\widehat{n}_0}^m\big(F_n^{-1}(v)\big), F_n^{-1}(u)\big]^+\Big)$$

is an approximation to the copula (3.9),

$$\lim_{n\to\infty}\lim_{m\to\infty} C_{m,n}^{\downarrow}(u, v; h) = \lim_{m\to\infty}\lim_{n\to\infty} C_{m,n}^{\downarrow}(u, v; h) = \lim_{m,n\to\infty} C_{m,n}^{\downarrow}(u, v; h),$$

and the common limit is $C_{X_t,X_{t+h}}^{\downarrow}(u, v)$ given by (3.9). *Furthermore, in all cases the limits are uniform in (u, v).*

Proof: If $T \in \mathcal{T}^{\uparrow}$, (i) follows from Theorem 5.1 by noticing that $K_h^{\downarrow} = \emptyset$, while if $T \in \mathcal{T}^{\downarrow}$, the opposite happens, namely, $K_h^{\uparrow} = \emptyset$, which implies (ii). ■

As an application of Theorem 5.1 and Lemma 3.3, we obtain the approximations for the copulas in the case where φ is an almost everywhere decreasing function.

Corollary 5.2. *Let $T \in \mathcal{T}^{\updownarrow}$ and μ_T be a T-invariant probability measure. Let $\varphi \in \mathcal{L}^1(\mu_T)$ be an almost everywhere decreasing function and let $\{X_t\}_{t=0}^{\infty}$ be the $\mathcal{T}_{\varphi}^{\updownarrow}$ associated process. Let μ_n be a sequence of measures converging weakly to μ_T and let F_n and F_n^{-1} be approximations to F_0 and F_0^{-1}, respectively, such that $F_n \to F_0$ and $F_n^{-1} \to F_0^{-1}$ uniformly. Suppose T has $s > 1$ nodes and, for $h > 1$, let $\{a_{h,i}^m\}_{i=0}^{s^h}$ be an approximation to the net related to T^h such that $a_{h,i}^m \to a_{h,i}$, for all $i = 0, \cdots, s^h$, as m goes to infinity. Also let $\{\mathcal{T}_{h,i}^m\}_{i=1}^{s^h}$ be an approximation to $\{\mathcal{T}_{h,i}\}_{i=1}^{s^h}$ and suppose $\mathcal{T}_{h,i}^m \to \mathcal{T}_{h,i}$ uniformly for all $i = 1, \cdots, s^h$. For all $(u, v) \in I^2$, let*

$$C_{m,n}^{\updownarrow *}(u, v; h) = u + v - 1 + C_{m,n}^{\updownarrow}(1-u, 1-v; h),$$

$$C_{m,n}^{\uparrow *}(u, v; h) = u + v - 1 + C_{m,n}^{\uparrow}(1-u, 1-v; h),$$

and

$$C_{m,n}^{\downarrow *}(u, v; h) = u + v - 1 + C_{m,n}^{\downarrow}(1-u, 1-v; h).$$

Then

$$\lim_{n\to\infty}\lim_{m\to\infty} C_{m,n}^{\updownarrow *}(u, v; h) = \lim_{m\to\infty}\lim_{n\to\infty} C_{m,n}^{\updownarrow *}(u, v; h) = \lim_{m,n\to\infty} C_{m,n}^{\updownarrow *}(u, v; h),$$

and the common limit is $C_{X_t,X_{t+h}}^{\updownarrow *}(u, v)$ given by (3.12),

$$\lim_{n\to\infty}\lim_{m\to\infty} C_{m,n}^{\uparrow *}(u, v; h) = \lim_{m\to\infty}\lim_{n\to\infty} C_{m,n}^{\uparrow *}(u, v; h) = \lim_{m,n\to\infty} C_{m,n}^{\uparrow *}(u, v; h),$$

and the common limit is $C_{X_t,X_{t+h}}^{\uparrow *}(u, v)$ given by (3.14),

$$\lim_{n\to\infty}\lim_{m\to\infty} C_{m,n}^{\downarrow *}(u, v; h) = \lim_{m\to\infty}\lim_{n\to\infty} C_{m,n}^{\downarrow *}(u, v; h) = \lim_{m,n\to\infty} C_{m,n}^{\downarrow *}(u, v; h),$$

and the common limit is $C_{X_t,X_{t+h}}^{\downarrow *}(u, v)$ given by (3.15). Furthermore, all the above limits are uniform in $(u, v) \in I^2$.

Proof: Immediate application of Theorem 5.1 and Lemma 3.3. ■

Remark 5.2. The uniform convergence of the approximations is a crucial hypothesis for the results in this subsection and cannot be dropped. Nevertheless, if the uniform convergence of any of the approximations for F_0, F_0^{-1} or $\mathcal{T}_{h,i}$ is violated, but the pointwise convergence to its target is maintained, the results of Theorem 5.1 and its corollaries will hold pointwisely instead of uniformly. The proof of Theorem 5.1 can be easily adapted to cover this case and the details are left to the reader. We observe that the approximations developed in Subsection 5.1 satisfy the conditions of the theorems presented in this section.

5.3 Random Variate Generation

In view of Proposition 3.6, obtaining a random sample from the copulas derived in Section 3 is a trivial task. Given $T \in \mathcal{T}^{\updownarrow}$ a transformation with $s > 1$ nodes, let μ_T be a T-invariant probability measure and $\varphi \in \mathcal{L}^1(\mu_T)$ be an almost everywhere increasing function. Let $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}_{\varphi}^{\updownarrow}$ process and let $\{a_{h,k}\}_{k=0}^{s^h}$ be the net associated to the nodes of T^h, for $h > 0$. Let $C_{X_t,X_{t+h}}^{\updownarrow}$ denote the copula associated to $\{X_t\}_{t\in\mathbb{N}}$. To obtain a random pair from $C_{X_t,X_{t+h}}^{\updownarrow}$, the following simple algorithm can be used:

1. Generate an uniform $(0,1)$ variate u.
2. Let k_0 be the index for which $u \in \big[F_0(a_{h,k_0-1}), F_0(a_{h,k_0})\big)$. Set
$$v := \begin{cases} \ell^{\uparrow}_{h,k_0}(u), & \text{if } \ k_0 \in K^{\uparrow}_h; \\ \ell^{\downarrow}_{h,k_0}(u), & \text{if } \ k_0 \in K^{\downarrow}_h. \end{cases}$$
3. The desired pair is (u,v).

Also notice that given two transformations $T_1, T_2 \in \mathcal{T}^{\updownarrow}$ such that both have the same discontinuity points and, in each node, they are either both increasing or both decreasing, then a sample from the copula related to one of the respective $\mathcal{T}^{\updownarrow}_{\varphi}$ processes cannot be distinguished from a sample to the other. This is so because from Proposition 3.6 both copulas have the same support, which, in this case, ultimately characterizes the sample.

Remark 5.3. By using a similar argument as in Proposition 3.6 associated with a cumbersome analysis of the possible cases, the support of the copulas derived in Section 5 can be obtained as well. Samples from those multidimensional copulas can be obtained in a similar fashion as in the bidimensional case.

6 Examples

In this section we present two examples of the copulas derived in the last sections. We concentrate ourselves on the bidimensional case, especially because there is no simple graphical representation of copulas in dimension higher than 2. The first presented example is the Manneville-Pomeau copula (MP copula for short) extracted from Lopes and Pumi (2013).

Example 6.1. For $s > 0$, consider the so-called Manneville Pomeau transformation $T_s : I \to I$, given by
$$T_s(x) = x + x^{1+s} (\mathrm{mod}\, 1).$$
Figure 1 shows the plot of the Manneville-Pomeau transformation for $s \in \{0.1, 0.5, 0.8\}$. If $s \in (0,1)$, there exists an absolutely continuous T_s-invariant probability measure, say μ_s, so that $T_s \in \mathcal{T}^{\uparrow}$. The MP copula is, therefore, given by (3.8), although a slightly different notation for n_0 is adopted in Lopes and Pumi (2013). No closed formula for μ_s is known, but it can be shown that μ_s is an SBR measure.

To approximate the MP copulas, the authors apply similar ideas to the ones presented here. To approximate μ_s, the authors use (5.2) and to approximate F_0^{-1}, a local linear interpolation function is applied. To approximate the net related to T_s^h and $\mathcal{T}_{h,k}$, the same scheme presented here, with a local linear interpolation function, is considered. For more details, see Lopes and Pumi (2013).

The next example is related to the well known Tent transformation (see Lopes and Lopes, 1998 and references therein).

Example 6.2. For $a \in (0,1)$, consider the piecewise linear transformation $T_a : I \to I$ given by
$$T_a(x) := \begin{cases} \dfrac{x}{a}, & \text{if } 0 \le x < a, \\ \dfrac{1-x}{1-a}, & \text{if } a \le x \le 1. \end{cases}$$
This is the so-called *Tent transformation*. Figure 6.1 presents the graphs of the Tent transformation for $a \in \{0.1, 0.5, 0.8\}$, where it can be seen that T_a has $s = 2$ full branches. Also, $T_a \in \mathcal{T}^{\updownarrow} \setminus (\mathcal{T}^{\uparrow} \bigcup \mathcal{T}^{\downarrow})$, since it can be shown that the Lebesgue measure in I^2 is a T_a-invariant

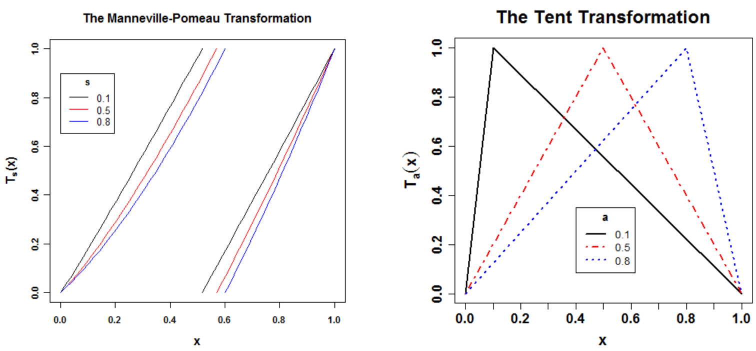


Figure 6.1: Plot of the Manneville-Pomeau transformation for $s \in \{0.1, 0.5, 0.8\}$ and the Tent transformation for $a \in \{0.1, 0.5, 0.8\}$.

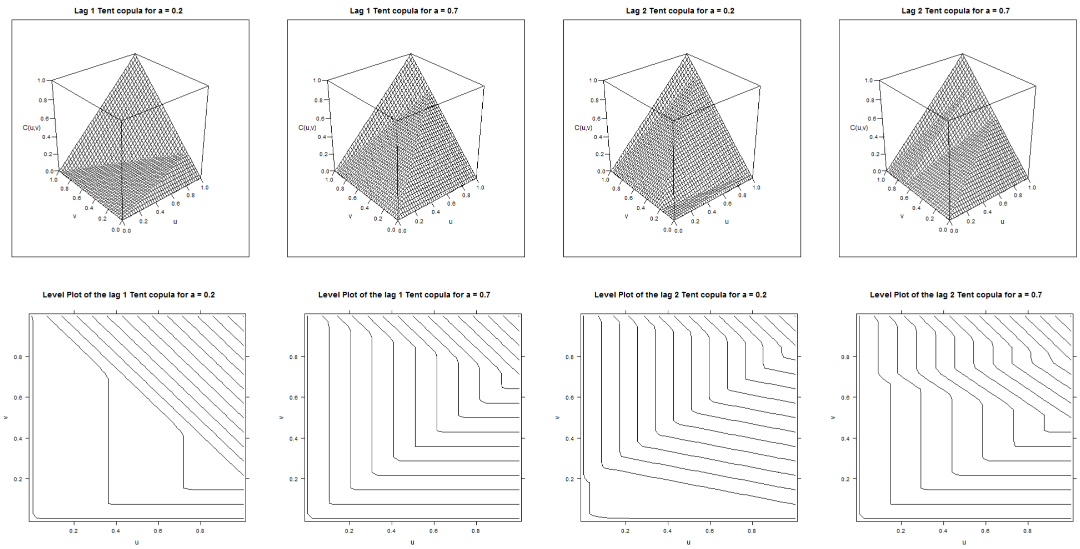


Figure 6.2: From left to right, three dimensional plots of the lag 1 and lag 2 Tent copula for $a \in \{0.2, 0.7\}$ (top panel) and respective level curves (bottom panel).

probability measure. This implies that $F_0(x) = x$ and $F_0^{-1}(y) = y$ everywhere. Let $\{X_t\}_{t\in\mathbb{N}}$ be the $\mathcal{T}_\varphi^{\updownarrow}$ process associated to T_a, with φ increasing almost everywhere. It is easy to see that, for any $h > 0$ and $t \geq 0$, the copula related to the pair (X_t, X_{t+h}), referred here as the *lag h Tent copula*, is given by

$$\begin{aligned} C_{X_t,X_{t+h}}(u,v) &= \sum_{k\in n_0^{\uparrow}} \big(\mathcal{T}_{h,k}(v) - a_{h,k-1}\big) + \big[\min\{u, \mathcal{T}_{h,n_0}(v)\} - a_{h,n_0-1}\big]\, \delta_{K_h^{\uparrow}}(n_0) + \\ &\quad + \sum_{k\in n_0^{\downarrow}} \big(a_{h,k} - \mathcal{T}_{h,k}(v)\big) + \max\{0, u - \mathcal{T}_{h,n_0}(v)\}\delta_{K_h^{\downarrow}}(n_0), \end{aligned}$$

where $K_h^{\uparrow}$ contains the odd numbers in $\{1, \cdots, 2^h\}$ and $K_h^{\downarrow}$ contains the even ones. Also notice that both, the net related to T^h and $\mathcal{T}_{h,k}$, can be determined exactly since, being each branch a linear function, any two points in the node suffice to give all information on the function to determine $\mathcal{T}_{h,k}$ and $a_{h,k}$. In this context, linear interpolation produces exact, instead of approximate, results.

Figure 6.2 shows the tree dimensional graphs of the lag 1 and 2 Tent copula for $a \in \{0.2, 0.7\}$ and its level curves. At the top panel of Figure 6.3, we present 500 sample points for the lag 1 and 2 Tent copula for $a \in \{0.2, 0.7\}$ and at the bottom panel, for the lag 7 Tent copula for $a \in \{0.2, 0.4, 0.5, 0.7\}$. Notice that, for small values of h, the sample resembles its support, but as h increases, it becomes hard to guess whether the sample came from a singular copula or from a continuous one.

The case $h = 1$ is very simple and the Tent copula can be easily calculated. In this case, the copula coincides with the one presented in Example 3.3 in Nelsen (2006). In the aforementioned example, the copula is derived by using a purely geometrical argument based only on the support, which coincides with the support of the lag 1 Tent copula. More details on the Tent transformation can also be found in Lopes and Lopes (1998) and references therein.

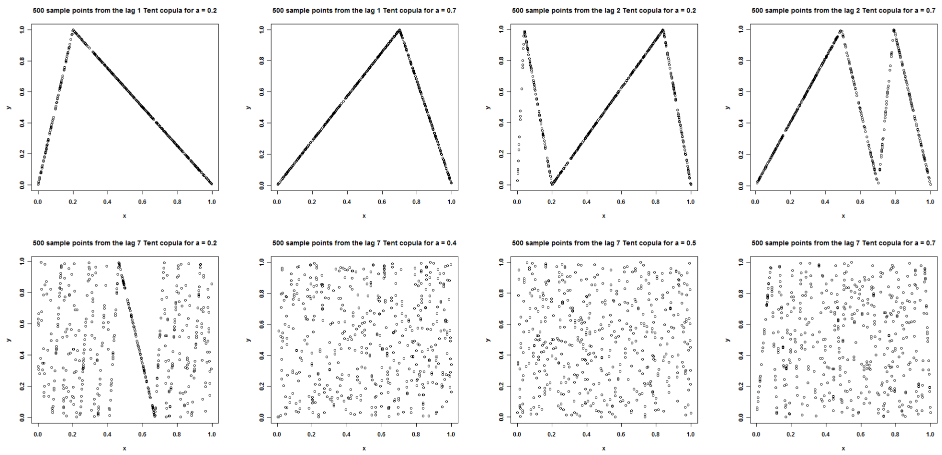


Figure 6.3: 500 sample points from the lag 1 and 2 Tent copula for $a \in \{0.2, 0.7\}$ (top panel) and the lag 7 Tent copula for $a \in \{0.2, 0.4, 0.5, 0.7\}$ (bottom panel).

7 Application

Let $T_{\boldsymbol{\theta}} \in \mathcal{T}^{\ddagger}$, for $\boldsymbol{\theta} \in S \subseteq \mathbb{R}^p$, $p \geq 1$, with $s > 1$ nodes. In this section we apply the general theory of Section 3 to the problem of estimating the parameter $\boldsymbol{\theta}$ based on a sample path of the associated $\mathcal{T}_{\varphi}^{\ddagger}$ process $\{X_t\}_{t\in\mathbb{N}}$, assuming that the parameter is identifiable through the knowledge of the discontinuity points of $T_{\boldsymbol{\theta}}$. The idea is to generalize the method described in Lopes and Pumi (2013), but the task will heavily depend on the transformation at hand. The general idea is as follows. According to Proposition 3.6, the support of the copula associated to a pair (X_t, X_{t+1}) from the $\mathcal{T}_{\varphi}^{\ddagger}$ process is the graph of a piecewise linear function joining consecutive points of the net and its image by $T_{\boldsymbol{\theta}}$ (equal to either 0 or 1). Let $L_{\boldsymbol{\theta}}$ denote this function. This implies that all points in a sample from the lag 1 copula lie on the graph of $L_{\boldsymbol{\theta}}$. Suppose for the moment that φ is the identity map and let $x_1, \cdots, x_N$ be a sample from the process X_t. Let $u_k := \big(F_0(x_k), F_0(x_{k+1})\big)$, $k = 1, \cdots, N-1$, where F_0 denotes the distribution of X_0 (for now, assume it is known). By Sklar's theorem, the sequence $\{u_k\}_{k=0}^{N-1}$ can be regarded as correlated sample from the lag 1 copula associated to the $\mathcal{T}_{\varphi}^{\ddagger}$ process. Often this simple situation makes an estimation possible.

Suppose that closed formulas for $F_0(\cdot;\boldsymbol{\theta})$ and $L_{\boldsymbol{\theta}}$, which may depend on $\boldsymbol{\theta}$, are available. Then the reasoning on the previous paragraph suggests the following optimization procedure to obtain an estimate $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$. Let $x_1, \cdots, x_N$ be a sample from the $\mathcal{T}_{\varphi}^{\ddagger}$ process and assume that we have at least two points in each branch of $T_{\boldsymbol{\theta}}$. Let $\mathscr{D}(\cdot,\cdot) : \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \to \mathbb{R}$ be a given function measuring the distance between two vectors in $\mathbb{R}^{N-1}$. With the notation on the beginning of Section 4, we define the estimate of $\boldsymbol{\theta}$ by

$$\hat{\boldsymbol{\theta}} := \underset{\boldsymbol{\theta}\in S}{\operatorname{argmin}} \Big\{ \mathscr{D}\big(F_0(x_{2:N};\boldsymbol{\theta}), L_{\boldsymbol{\theta}}(x_{1:N-1})\big) \Big\}. \tag{7.9}$$

Usual choices of $\mathscr{D}$ are $\mathscr{D}(\boldsymbol{x},\boldsymbol{y}) = \sum_{k=1}^{N-1}(x_k - y_k)^2$ and $\mathscr{D}(\boldsymbol{x},\boldsymbol{y}) = \sum_{k=1}^{N-1}|x_k - y_k|$, where $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^{N-1}$. The optimization procedure (7.9) is justified by noticing that for the correct $\boldsymbol{\theta}$, $\big(F_0(x_k;\boldsymbol{\theta}), F_0(x_{k+1};\boldsymbol{\theta})\big) = \big(F_0(x_k;\boldsymbol{\theta}), L_{\boldsymbol{\theta}}\big(F_0(x_{k+1};\boldsymbol{\theta})\big)\big)$, for all k, while for a misspecified $\boldsymbol{\theta}$, the equality does not hold. Most of the times, however, the particular form of F_0 is not known and often the net cannot be obtained analytically. This is the case of the Manneville-Pomeau transformation, for instance. A solution for this problem is to use approximations for F_0 and $\{a_k\}_{k=0}^s$ in the spirit of Subsection 5.1. However, an optimization procedure such as (7.9) performed by using the approximations of Subsection 5.1 may be computationally too expensive for all practical purposes, so we shall seek for another faster estimation method.

In practice many different situations may occur. For instance, if T_a is the Tent transformation in Example 6.2, then F_0 is the identity map, so that $u_k = (x_k, x_{k+1})$ in the notation of the previous paragraph. This means that any two points u_k in the same branch are sufficient to obtain the parameter a.

Another method can be devised upon noticing that the net of $T_{\boldsymbol{\theta}}$ and the net of $L_{\boldsymbol{\theta}}$ (whose graph is the support of the copula C associated to the $\mathcal{T}_\varphi^\updownarrow$ process) are the same. Let $v_k := (x_k, x_{k+1})$, $k = 1,\cdots,N-1$. By Sklar's theorem, the points $\{v_k\}_{k=1}^{N-1}$ lie on the graph of $T_{\boldsymbol{\theta}}$. At this point, it could be possible in some cases to obtain $\boldsymbol{\theta}$ by a more direct argument (see Example 7.1) or by applying an argument such as the preceding paragraphs. That is, for $\mathscr{D}$ a function as above, $\boldsymbol{\theta}$ can be obtained by performing the optimization procedure

$$\hat{\boldsymbol{\theta}} := \underset{\boldsymbol{\theta}\in S}{\operatorname{argmin}}\Big\{\mathscr{D}\big(x_{2:N}, T_{\boldsymbol{\theta}}(x_{1:N-1})\big)\Big\}.$$

The methods above may not be applicable or may fail, especially for high values of s. In this case another computationally fast method is obtained by adapting the argument in Lopes and Pumi (2013). Let $T_{\boldsymbol{\theta}} \in \mathcal{T}^\updownarrow$, with $s > 1$ branches and $\boldsymbol{\theta} := (\theta_1,\cdots,\theta_p) \in S \subseteq \mathbb{R}^p$, for $1 \le p \le s-1$. Let $\{a_k\}_{k=0}^s$ denote the net associated to $T_{\boldsymbol{\theta}}$. Suppose that $\boldsymbol{\theta}$ can be uniquely determined by the knowledge of the net $\{a_k\}_{k=0}^s$ in the sense that, given $\{a_k\}_{k=0}^s$, $\boldsymbol{\theta}$ can be computed. That is, $\boldsymbol{\theta} := f(a_0,\cdots,a_s)$, where $f : I^{s+1} \to S$ is a known smooth function. Notice that we allow for θ_i to be obtainable from two or more different a_k's as long as the value of θ_i agree in all cases. Let $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}_\varphi^\updownarrow$ process, assuming that φ is the identity map. The goal is to estimate the parameter $\boldsymbol{\theta}$ based on a sample $x_1,\cdots,x_N$ from the process X_t.

Given that $T_{\boldsymbol{\theta}} \in \mathcal{T}^\updownarrow$, F_0 is generally smooth so that, near $\{a_k\}_{k=0}^s$, $T_{\boldsymbol{\theta}}$ should locally behave like a linear function. With this in mind, the estimation of $\boldsymbol{\theta}$ can be performed as follows. Let $\{I_k\}_{k=1}^s$ be the nodes relative to $T_{\boldsymbol{\theta}}$. Let $P_k := \{x_{m+1} : x_m \in I_k, m = 1,\cdots,N-1\}$, $k = 1,\cdots,s$. Notice that, if $x_m \in P_k$, v_m lies on the graph of the k-th branch of $T_{\boldsymbol{\theta}}$. Set $x_k^+ := \max\{x : x \in P_k\}$ and $x_k^- := \min\{x : x \in P_k\}$. Let

$$\tilde{a}_k(y) := \frac{y-\beta_k}{\alpha_k}, \quad \text{where} \quad \alpha_k := \tfrac{T_{\boldsymbol{\theta}}(x_k^+)-T_{\boldsymbol{\theta}}(x_k^-)}{x_k^+-x_k^-} \quad \text{and} \quad \beta_k := T_{\boldsymbol{\theta}}(x_k^+) - \alpha_k x_k^+.$$

Notice that for each $y \in \mathbb{R}$, $\tilde{a}_k(y)$ is the inverse image of y by the linear function connecting $\big(x_k^-, T_{\boldsymbol{\theta}}(x_k^-)\big)$ and $\big(x_k^+, T_{\boldsymbol{\theta}}(x_k^+)\big)$. For $k \in \{1,\cdots,s-1\}$, we define the estimator $\hat{a}_k$ of a_k by

$$\hat{a}_k := \begin{cases} \tilde{a}_k(1), & \text{if } T_{\boldsymbol{\theta}} \text{ is increasing in } I_k,\\ \tilde{a}_k(0), & \text{if } T_{\boldsymbol{\theta}} \text{ is decreasing in } I_k.\end{cases} \tag{7.10}$$

Once we have the estimates $\{\hat{a}_k\}_{k=1}^{s-1}$ (recall that $a_0 = 0$ and $a_s = 1$), we obtain the estimate $\hat{\boldsymbol{\theta}} := f(0,\hat{a}_1,\cdots,\hat{a}_{s-1},1)$.

Remark 7.1. Notice that by inverting the roles of $\tilde{a}_k(0)$ and $\tilde{a}_k(1)$ in (7.10) we obtain an estimate of a_{k-1}. That is, for $k \in \{2, \cdots, s\}$, define

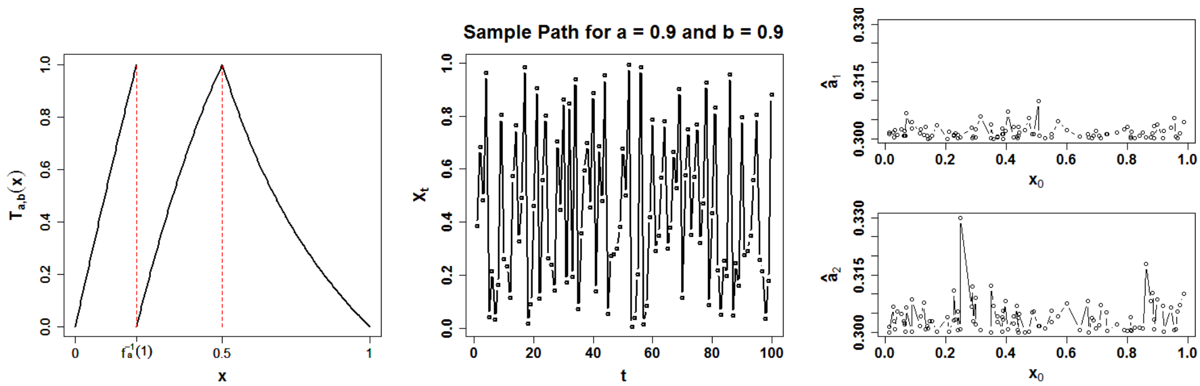
$$\hat{a}_{k-1} := \begin{cases} \tilde{a}_k(0), & \text{if } T_{\boldsymbol{\theta}} \text{ is increasing in } I_k, \\ \tilde{a}_k(1), & \text{if } T_{\boldsymbol{\theta}} \text{ is decreasing in } I_k. \end{cases} \tag{7.11}$$

Thus, for each $k = 2, \cdots, s-1$ we obtain an estimate of a_k from I_k by using (7.10) and another one from I_{k+1} by using (7.11). Of course, the net related to $T_{\boldsymbol{\theta}}$ is unique and we assume that $\boldsymbol{\theta}$ is uniquely determined by the knowledge of the net, but in practice, the estimates (7.10) and (7.11) usually do not agree. This implies that the estimated value of $\boldsymbol{\theta}$ depends on which estimator we apply. Hence, the question which one provides better results is a valid one. The answer is intuitive: the "best" one is usually the one obtained from the node which is closer to a straight line and for which the points v_k's are closer to $(a_k, 0)$ and $(a_{k+1}, 1)$. In this case the line connecting $\big(x_k^-, T_{\boldsymbol{\theta}}(x_k^-)\big)$ and $\big(x_k^+, T_{\boldsymbol{\theta}}(x_k^+)\big)$ is "closer" to the respective branch of the copula's support.

Example 7.1. Let $a \in [0,1]$ and $b \in (0,1)$ and consider the map

$$T_{a,b}(x) := \begin{cases} f_a(x)(\text{mod}1), & \text{if } 0 \le x < \frac{1}{2}, \\ \frac{(2-b)(1-x)}{1-b+bx}, & \text{if } \frac{1}{2} \le x \le 1, \end{cases} \tag{7.12}$$

where $f_a(x) := 2[2 - 0.6a\sin(2.1a)]x + 0.6a\sin(4.2ax)$. The map $T_{a,b}$ has 3 full branches, two of them are increasing and one is decreasing. Figure 7.4(a) presents the typical graph of $T_{a,b}$. It is easy to see that $T_{a,b}$ is uniformly expanding and it can be shown that there exists an absolutely continuous $T_{a,b}$-invariant probability measure (cf. Theorem 1 and Remark 1 in Pianigiani, 1980) and so $T_{a,b} \in \mathcal{T}^{\updownarrow}$. Let $\{X_t\}_{t\in\mathbb{N}}$ be the associated $\mathcal{T}_{\varphi}^{\updownarrow}$ process. A sample of size 100 from the $\mathcal{T}_{\varphi}^{\updownarrow}$ process with $a = b = 0.9$, for φ the identity map is shown in Figure 7.4(b).



(a) (b) (c)

Figure 7.4: (a) A typical graph of $T_{a,b}$; (b) A sample of size 100 from the $\mathcal{T}_{\varphi}^{\updownarrow}$ process with $a = b = 0.9$ for φ the identity map and $X_0 = 0.385969$; (c) Behavior of the estimators $\hat{a}_1$ and $\hat{a}_2$ as a function of the initial point x_0. The parameters are $a = 0.3$ and $b = 0.5$ in all cases.

To exemplify the estimation procedure developed in this section, we perform the following Monte Carlo simulation study. For 100 randomly chosen initial points[5], we simulate a sample of size 100 from the $\mathcal{T}_{\varphi}^{\updownarrow}$ process $\{X_t\}_{t\in\mathbb{N}}$ for all combinations of parameters a and b ranging over the set $\{0.1, 0.3, 0.5, 0.7, 0.9\}$. For simplicity, let $\{d_0, \cdots, d_3\}$ denote the net of $T_{a,b}$. In this case $d_0 = 0$, $d_2 = 0.5$ and $d_3 = 1$, so that d_1 is the only discontinuity to be estimated. This estimation

[5] in theory, each point should be irrational, but since computers can only express rational numbers, these should be chosen "as irrational as possible" in the sense to use all possible decimal points allowed under the machine's precision to express an irrational number

Table 7.1: Simulation results based on samples of size 100 of the $\mathcal{T}_\varphi^{\updownarrow}$ process associated to (7.12) for 100 randomly chosen initial points and parameters a and b ranging over the set $\{0.1, 0.3, 0.5, 0.7, 0.9\}$. Presented are the estimated values and the respective mean square error ($\times 10^{-4}$) in parenthesis.

b	0.1		0.3		0.5		0.7		0.9	
a	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_1$	$\hat{a}_2$
0.1	0.1005 (0.0045)	0.1011 (0.0251)	0.1007 (0.0089)	0.1012 (0.0303)	0.1005 (0.0055)	0.1011 (0.0224)	0.1005 (0.0054)	0.1011 (0.0248)	0.1005 (0.0039)	0.1011 (0.0241)
0.3	0.3018 (0.0589)	0.3040 (0.2491)	0.3019 (0.0690)	0.3045 (0.3625)	0.3017 (0.0585)	0.3039 (0.3394)	0.3016 (0.0486)	0.3033 (0.2056)	0.3015 (0.0439)	0.3035 (0.2462)
0.5	0.5033 (0.2031)	0.5064 (0.7755)	0.5027 (0.1332)	0.5072 (1.1065)	0.5024 (0.0924)	0.5054 (0.5878)	0.5027 (0.1215)	0.5054 (0.6567)	0.5030 (0.1528)	0.5064 (0.9126)
0.7	0.7046 (0.3845)	0.7130 (2.9859)	0.7039 (0.2515)	0.7097 (1.8811)	0.7042 (0.3369)	0.7099 (2.2650)	0.7040 (0.2767)	0.7087 (1.3976)	0.7044 (0.3638)	0.7090 (1.5546)
0.9	0.9051 (0.4326)	0.9185 (5.9339)	0.9043 (0.3095)	0.9172 (6.3317)	0.9049 (0.4276)	0.9174 (6.8437)	0.9050 (0.4598)	0.9182 (7.0888)	0.9049 (0.5110)	0.9162 (4.2946)

is performed by using the two methods (7.10) based on the first branch and (7.11) based on the second branch, the estimators of d_1 being denoted by $\hat{d}_1^{(1)}$ and $\hat{d}_1^{(2)}$, respectively. From $\hat{d}_1^{(1)}$ and $\hat{d}_1^{(2)}$, we set $\hat{a}_i := f_a^{-1}(d_1^{(i)})$, $i = 1, 2$, where the inverse of f_a is obtained numerically. The parameter b needs no estimation since it can be exactly calculated by noticing that for a pair (x_k, x_{k+1}), with $0.5 < x_k < 1$, b is exactly obtained by the formula $b = \frac{2(1-x_k)-x_{k+1}}{(1-x_k)(1-x_{k+1})}$.

The simulation results are reported in Table 7.1, where the mean estimated value along with its mean square error (values in parenthesis $\times 10^{-4}$) are presented. The overall performance of $\hat{a}_1$ and $\hat{a}_2$ are very good with small bias and small variability. As expected (see Remark 7.1 and Figure 7.4(a)), $\hat{a}_1$ outperforms $\hat{a}_2$ since the first branch of $T_{a,b}$ is "closer" to linear than the second one. Both estimators always overestimate the true parameter. The behavior of the estimators $\hat{a}_1$ and $\hat{a}_2$ as a function of the initial point x_0 for $a = 0.3$ and $b = 0.5$ are presented in Figure 7.4(c). Also notice that the particular value of the parameter b does not significantly affects the estimation of parameter a.

8 Conclusions

In this work we study the copulas related to pairs and vectors of random variables coming from a class of stochastic processes defined in terms of iterations of a certain smooth piecewise monotonic transformation of the interval to some initial random variable. More specifically, we study the copulas related to random variables coming from a stochastic process defined as $X_t = \varphi\big(T^t(U_0)\big)$ for T a piecewise monotonic transformation of the interval, U_0 an initial random variable distributed according to a T-invariant probability measure and a smooth function $\varphi : [0,1] \to \mathbb{R}$, taken to be monotonic. We show that the copulas depend only on the lag h between components and we derived formulas and properties for both, the joint distribution function and the copulas for pairs of random variables coming from the process $\{X_t\}_{t\in\mathbb{N}}$. As expected, the copulas heavily depend on the T-invariant probability measure. The multidimensional case is similar to the bidimensional one so we follow the same agenda as in the latter and we are able to show analogous results. We notice the similarities among the results of Sections 3 and 4 when φ is increasing. For φ decreasing, the multidimensional case poses some difficulties which ultimately lead to more complicated formulas compared to the bidimensional case. We also discuss random variate generation and approximations for the copulas derived on Section 3.

The general theory is applied to the problem of parametric estimation in $\mathcal{T}_\varphi^\ddagger$ processes. Examples and a simple Monte Carlo study are also provided.

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